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1884.

Calculus of Direction and Position.

By E. W. Hyde, *Cincinnati.*

It appears to me to be obvious that the calculus of directed quantities in some form ought to and will come more and more into use in all the operations of Geometry and Mechanics, owing to its peculiar fitness for expressing the relations and conditions of the quantities discussed in these branches of science. It is therefore important that the *best* and most *natural* system should be employed, and taught to the students who are to form the next generation of mathematicians and scientific men.

I propose in the present paper to give briefly the fundamental idea and principles of Hamilton's system, or "Quaternions," and of Grassmann's, called by him "Die Ausdehnungslehre," showing their points of difference, and what seem to me to be strong reasons why the system of Grassmann is far preferable.

The title of this paper indicates at once a most important difference between the two systems, for Quaternions is a calculus of magnitude and direction *only*, while the "Ausdehnungslehre" is a calculus of magnitude, direction and *position.* This is precisely what is required for Mechanics, for a force is fully determined only when these three things are known concerning it. It also greatly facilitates many operations of Geometry, as will appear in the sequel.

Hamilton's ruling idea in forming his system was *rotation*, and his versors are really $\sqrt{-1}$, endowed with a directed quality so as to turn the vector operated on about some particular *axis*. The numerical quantity $\sqrt{-1}$ may be called an *undirected* versor.

The *addition* and *subtraction* of directed quantities was understood before the invention of Quaternions, the peculiarity of which depends on its method of *multiplication.*

The basis of the whole theory may be given in a few words as follows:

Taking I, J, K as three mutually perpendicular unit vectors, and i, j, k as operators or versors that turn J into K, K into I and I into J respectively, it is shown that

$$\left\{\begin{matrix} ij = k \\ iJ = K \end{matrix}\right\}, \quad \left\{\begin{matrix} ji = -k \\ jI = -K \end{matrix}\right\} \text{ etc.}$$

(*not*, however, that $IJ=K$), and then it is *assumed* that it is permissible to take i, j, k as *identical* with I, J, K, which is certainly true, the only question being whether it is on the whole *best* to do so.

Tait says (Art. 72, Tait's Quat.): "Now the meanings we have assigned to i, j, k, are quite independent of, and not inconsistent with, those assigned to I, J, K. And it is superfluous to use two sets of characters where one will suffice. Hence it appears that i, j, k may be substituted for I, J, K; in other words, *a unit-vector when employed as a factor may be considered as a quadrantal versor whose plane is perpendicular to the vector.* This is one of the main elements of the singular simplicity of the quaternion calculus."

This last statement I entirely disagree with.

Again, in Art. 64, Tait says: "We shall content ourselves at present with an assumption, which will be shown to lead to consistent results; but at the end of the chapter we shall show that no other assumption is possible, following for this purpose a very curious quasi-metaphysical speculation of Hamilton."

The statement "that no other assumption is possible" I deny, and I propose to show that, though Hamilton's reasoning is correct, the deductions therefrom are unwarranted.

The "speculation" referred to above, as given in abridged form by Tait, is as follows (see Art. 93, Tait's Quat.): "Suppose that no direction in space is preëminent, and that the product of two vectors is something that has quantity, so as to vary in amount if the factors are changed, and to have its sign changed if that of one of them is reversed; if the vectors be parallel, their product cannot be, in whole or in part, a vector *inclined* to them, for there is nothing to determine the direction in which it must lie. It cannot be a vector *parallel* to them; for by changing the sign of both factors the product is unchanged, whereas, as the whole system has been reversed, the product vector ought to have been reversed. Hence it must be a number. Again, the product of two perpendicular vectors cannot be wholly or partly a number, because on inverting one of them the sign of that number ought to change; but inverting one of them is simply equivalent to a rotation through two right angles about the other, and (from the symmetry of space) ought to leave the number unchanged. Hence the product of two perpendicular vectors must be a vector, and a simple extension of the same reasoning shows that it must be perpendicular to each of the factors."

Now the reasoning as to the product of $\parallel$ vectors, *i. e.* that it is scalar, is perfectly correct, but it does not follow that this product must be — the product of their tensors, as in quaternions; it may, for instance, be zero, as will be shown hereafter.

The reasoning also with regard to the product of two $\perp$ vectors is correct, if we write *vector quantity* or *directed quantity* instead of vector in the last sentence. But a plane area is a quantity having magnitude and direction as well as a portion of a right line, so that there is nothing in the reasoning which precludes ij from meaning a square unit of plane area $\parallel$ to i and j, which certainly appears a more natural signification than that ij should be *equal* to k, a unit vector $\perp$ to i and j. From this assumption it follows as above, that $ij = k$ and also that $i/j = -ij = -k$, *i. e.* the *ratio* of two quantities is the same thing as their *product* except as to sign. To be sure we may say that these are *units*, and we have the analogy that $1/1 = 1 \times 1$; but they, *i. e.* vectors, are *geometric* and *directed* units, and such a relation appears to me to upset all one's preconceived ideas of geometric quantities without any corresponding advantage. If, in the eq. $1/1 = 1 \times 1$, 1 be taken as a unit of *length*, then the members of the equation have evidently not the same meaning, $1/1$ being merely a numerical quantity while 1×1 is a unit of *area*, it being a fundamental geometric conception that the product of a length by a length is an *area*, that of a length by an area a volume, while the ratio of two quantities of the same order as that of a length to a length is a mere number of the order zero. In quaternions however we have the remarkable result that the product of a length by a length is not merely represented by, but actually *equal* to a length $\perp$ to the plane of the two.

Of course this arises from the double function of the vector as used in quaternions, it being not only something endowed with magnitude and direction, but also possessing the properties of the versor $\sqrt{-1}$.

The combination of these different functions in the vector renders the product of two vectors which are neither $\parallel$ nor $\perp$ to each other necessarily a *complex* quantity, having a scalar and a vector part corresponding to the real and imaginary parts of the ordinary complex $a + b\sqrt{-1}$, thus making a thing which should be simple just the opposite.

It seems to me that quaternions proper, *i. e.* these complex quantities, are practically of little use. In nearly all the applications to geometry and mech-

anics, scalars or vectors are used *separately*. For the special uses to which the complex $a + b\sqrt{-1}$ is put, the directed quality is not needed.

Another point about this system appears objectionable to my mind, viz. that we must *necessarily* work in space of three dimensions. Even when nominally treating plane geometry, the product of any two vectors in the plane is a vector $\perp$ to it, and we are therefore really treating space.

We will now consider Grassmann's system, giving first the way in which he was led to his method of multiplying directed quantities, as stated by himself in the preface to his first book, published in 1844, and then giving a brief account of the whole system.

In the above-mentioned preface he first states how he was led to the addition and subtraction of vectors (*strecken*), and afterwards to their multiplication. The addition and subtraction being precisely as in quaternions, we will not here consider them.

Grassmann arrived at his conception of the product of two *directed* lines from a consideration of the geometric meaning of the product of two *undirected* lines, viz. the rectangle having these two lines for two of its conterminous sides. It would follow at once from analogy that the product of two directed lines at right angles should be the same rectangle as before, endowed with the additional property of *direction*, *i. e.* it would be a plane area $\parallel$ to the two given vectors and numerically equal to the product of their lengths. A further extension of the analogy would indicate that the product of *any* two directed lines should be the area of the parallelogram of which these lines are two adjacent sides, or an equal parallelogram *parallel* to this; and similarly the product of *three* vectors should be the volume of the parallelopiped of which they form three conterminous edges. These conceptions were found to work consistently, and it appeared that this species of multiplication agreed with ordinary multiplication in being subject to the associative and distributive laws, but different from it in that it did *not* obey the commutative law.

Thus, if a, b, c are three non-co-planar vectors, we have $abc = a.bc = ab.c$, $ab + ac = a(b + c)$, but *not* $ab = ba$. Instead of the last we have

$$ab = -ba \text{ or } ab + ba = 0.$$

It follows at once from this non-commutative law, as well as from the meaning of the product of two vectors given above, that $aa = 0$.

Similarly $aab = 0$, $aba = 0$, etc.

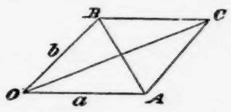
These are the laws of what Grassmann calls "outer multiplication," and I think their simplicity will be acknowledged as compared with the multiplication of vectors in quaternions.

The conception formed by Grassmann of the product ab was that the vector a generates the area ab by moving parallel to itself along the vector b as a directrix from its initial to its final point. This idea of generation by a moving point, line or surface led to his use of the term "ausdehnung" or "extension" as descriptive of his system.

We will now give a brief sketch of Grassmann's method of treating points and vectors.

A point is some position in space with a value attached to it which we may call its *weight* (using the term somewhat as in the theory of least squares), and which may be positive or negative. We may use a letter, as A, to represent a unit point, *i. e.* a point of unit weight, and then a point whose weight is m will be mA.

Two unit points can differ only in *position*, *i. e.* by a certain length of straight line in a certain direction, or, in other words, by a *vector*. Hence if A and B are two unit points, and a a vector drawn from A to B, we have $B - A = a$. Also at once if $B - O = b$ and $A - O = a$



$$B - A = B - O - A + O = B - O - (A - O) = b - a$$

and $\quad a + b = A - O + B - O = A - O + C - A = C - O.$

Since points and vectors can thus be added and subtracted they must be quantities of the same kind, and in fact a vector is a point of weight zero situated at infinity, as will presently appear.

The sum of two points as mA and nB must be something of the same kind, and therefore a *point*, and its weight should evidently be equal to the sum of the weights of the two points, *i. e.* $m + n$: hence write

$$mA + nB = (m + n)\,S,$$

S being a unit point. To find the position of S subtract $(m + n)O$ from both sides of the above equation, O being any unit point whatever.

$$\therefore \quad m(A - O) + n(B - O) = (m + n)(S - O), \text{ or}$$

$$S - O = \frac{m(A - O) + n(B - O)}{m + n};$$

by which S may be constructed. This equation shows that the sum of two points is their *mean point*. Or it may be put thus: If m and n are regarded as parallel forces acting at A and B, then S is the centre of these parallel forces.

The same holds for any number of points.

Next suppose $m + n = 0$, or $m = -n$; then we have

$$mA + nB = n(B - A) = 0 . S = 0 . (S - O).$$

To satisfy these equations we must either have $m = n = 0$, or $A = B$, *i. e.* A the same point as B, or else $S - O = \infty$, *i. e.* S situated at ∞; thus it appears as stated above that a vector is a point at ∞.

Passing now to multiplication, we will start with two or three definitions, and a statement of the geometric meaning of outer multiplication more general than that previously given, and from a different point of view.

Two posited quantities differ in position when they have no point in common, and not otherwise.

According to this definition, parallel right lines or a plane and parallel line do not differ in position, as in each case there is a common point at ∞.

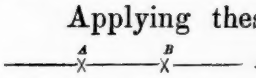
A *point* is said to be of the *first order*, the product of *two* points of the *second order*, etc. (*order* corresponds to Grassmann's *stufe*). The locus of all points which can be expressed in terms of n given points is called a *region* of the n^{th} order. Thus a plane is a region of the third order; space of three dimensions a region of the fourth order.

The outer product of two posited quantities which differ in position is some multiple of the connecting figure.

If two posited quantities do *not* differ in position, the connecting figure will be zero, and therefore the product zero. This is true whenever the sum of the orders of the two factors is not greater than the order of the region under consideration. Or, in other words, whenever the two factors are such that it is *possible* for them to differ in position in the region under consideration.

For instance, if we are considering space, or the region of the fourth order, two right lines *can* differ in position, while if we are dealing with a *plane* region they *cannot.*

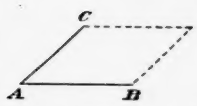
The outer product of two posited quantities which cannot differ in position in the region under consideration is the common figure multiplied by a scalar quantity.

Applying these principles we have for the product of two points that 

AB is the line from A to B; that is, it is a portion of the right line fixed by the two points A and B, whose length is equal to the distance from A to B. It may be situated anywhere on the line. These are exactly the conditions that completely determine a *force.* AB differs from $B - A$, in that the latter may be any *parallel* line of the same length.

If $B - A = a$ we have $Aa = A(B - A) = AB$, so that multiplying a vector by a point fixes its position by making it pass through the point.

Grassmann gives the name "Linientheil" to the product of two points, or a point and a vector, but it appears to me that an appropriate and expressive name in English would be *point-vector*.



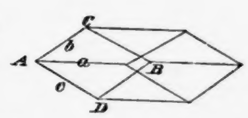
The product ABC is *twice* the connecting triangle, *i. e.* the parallelogram of which A, B, C are three corners.

That the product should be twice the connecting figure appears from the previous way of looking at outer multiplication: for AB is generated by a point in moving from A to B, and ABC is similarly generated by a right line of length AB moving parallel to itself from A to C.

We have, if $B - A = a$ and $C - A = b$,

$$ABC = AB(C - A) = ABb = A(B - A)b = Aab.$$

I have in my lectures on this subject called the product of two vectors, as ab, a *plane-vector*, and the product $ABC = ABb = Aab$ a *point-plane-vector;* the difference being that the latter is fixed in position in so far as that it lies in a fixed plane, while the former may lie in any parallel plane.



The product $ABCD$ is six times the connecting tetrahedron, that is, the parallelopiped of which A, B, C, D are four vertices.

As before we may write $ABCD = ABCc = ABbc = Aabc$, but in this case each of these is equal to abc, since there is only *one* space of three dimensions.

As these products can have no direct or posited quality they are *scalars*.

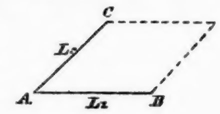
In the region of the fourth order, or space, we have

$$ABCD = ABC.D = A.BCD = AB.CD = L_1 L_2,$$

if we put $L_1 = AB$ and $L_2 = CD$, so that the product of two point-vectors is six times the tetrahedron of which they are opposite edges.

In dealing with a *plane region* (as in plane geometry) all the rest of space except this plane may be regarded for the time being as non-existent, so that plane-vectors and point-plane-vectors lose their property of direction and can differ only in magnitude and sign; hence they are *scalar* quantities, a fact which considerably facilitates the application of this method to plane geometry. The difference between a plane-vector and a point-plane-vector also disappears, since but *one* plane is under consideration, so that we have $Aab = ab = TaTb \sin{}^{b}_{a}$.

We will now look at the product of some quantities which *cannot* differ in position. In a *plane-region,* if L_1 and L_2 are two point-vectors, they cannot differ in position. Let A be the common point of L_1 and L_2, and let B and C be so taken that $AB = L_1$ and $AC = L_2$; then $L_1 L_2 = AB . AC = ABC . A$: that is, the product is the common point multiplied by the scalar ABC, which accords with the definition previously given.



In space, let $L = AB$ and $P = ACD$; then $LP = AB.ACD = ABCD.A =$ the common point multiplied by the scalar $ABCD$.

Similarly, if $P_1 = ABC$ and $P_2 = ABD$, $P_1 P_2 = ABC.ABD = ABCD.AB$ $=$ the common point-vector multiplied by the scalar $ABCD$.

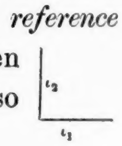
Also $ab.ac = abc.a$, *i. e.* the product of two plane-vectors is a vector $\|$ to each of them multiplied by a volume.

In most cases in the use of these products in geometry the scalar coefficients may be disregarded, and we may consider AB as the line passing through A and B of indefinite length, ABC as the plane through A, B and C, $L_1 L_2$ (in a plane region) as the common point only, $P_1 P_2$ as the common line of these two planes, etc.

We have thus a simple and complete system of geometric multiplication, in which every product has a clear and definite meaning to be seen at a glance, contrasting thus strongly with many quaternion expressions. What geometric meaning, for instance, can be easily assigned to the expression $V.\alpha\beta\gamma$? Of course there *is* such a meaning which can be gotten at with labor enough, but it is anything but an *evident* meaning. Or let a, b, c, d be four vectors and compare the two equivalent expressions $ab.cd$ and $V.Vab\,Vcd$; the meaning of the first is seen at a glance, while that of the second requires a considerable mental operation to determine it, to say nothing of the additional labor of writing it.

To complete this brief review of Grassmann's system we have only to consider what he denominates "inner multiplication."

In a plane region any vector may be expressed in terms of two given vectors; if these be unit vectors at right angles we have a *unit normal reference system.* In such a system let the two reference vectors be ι_1 and ι_2, then ι_2 is the *complement* of ι_1, written $/\iota_1$, and $-\iota_1 = /\iota_2$, the sign being so taken that $\iota_1/\iota_1 = \iota_1\iota_2 = 1$ and $\iota_2/\iota_2 = -\iota_2\iota_1 = 1$.

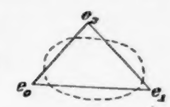


If a and b be any vectors we easily find that $/a$ is $\perp$ to a, and that $a/b = TaTb \cos_a^b$; so that $a/b = 0$ is the condition of perpendicularity between a and b.

Grassmann calls a/b the *inner* product of a and b, but it is also the *outer* product of a and $/b$, which is generally the most convenient way to regard it. We have also $a/a = T^2 a \cos 0 = T^2 a = a^{\underline{2}}$, the last form being called the *inner square* of a, and written with the exponent underscored to distinguish from $a^2 = aa$ which is always zero.

It will be seen that a/b is the same as Hamilton's Sab except that it is *positive* for values of the angle between a and b less than 90°, a manifest advantage.

Similarly all the points in a plane region can be expressed in terms of three given points e_0, e_1, e_2, and these may be so taken that $e_0 e_1 e_2 = 1$, then



$$/e_0 = e_1 e_2, \text{ so that } e_0/e_0 = e_0 e_1 e_2 = 1,$$
$$/e_1 = e_2 e_0, \quad \text{"} \quad e_1/e_1 = e_1 e_2 e_0 = e_0 e_1 e_2 = 1,$$
$$/e_2 = e_0 e_1, \quad \text{"} \quad e_2/e_2 = e_2 e_0 e_1 = e_0 e_1 e_2 = 1.$$

If an ellipse be so drawn that $e_1 e_2$ is in it the anti-polar of e_0, $e_2 e_0$ the anti-polar of e_1, and $e_0 e_1$ the anti-polar of e_2; then, if p be any point whatever, $/p$ is its anti-polar in this same ellipse.

In *space* we have similarly, if ι_1, ι_2, ι_3 form a *unit normal system*,

$$/\iota_1 = \iota_2 \iota_3, \text{ so that } \iota_1/\iota_1 = \iota_1 \iota_2 \iota_3 = 1, \text{ etc., etc.,}$$

$/a$ is now a *plane-vector* $\perp$ to a, and a/b has the same meaning as before.

In a *point* system e_0, e_1, e_2, e_3 to which all points in space may be referred, and which may be so taken that $e_0 e_1 e_2 e_3 = 1$, we have

$$/e_0 = \quad e_1 e_2 e_3, \text{ so that } e_0/e_0 = \quad e_0 e_1 e_2 e_3 = 1,$$
$$/e_1 = - e_2 e_3 e_0, \quad \text{"} \quad e_1/e_1 = - e_1 e_2 e_3 e_0 = e_0 e_1 e_2 e_3 = 1, \text{ etc., etc.}$$

Any point p is the anti-pole of its complement $/p$ in an ellipsoid, so taken that in it $/e_0$ is the anti-polar plane of e_0, etc.

One of the great advantages of this system over quaternions is, that while better adapted for treating geometry by the use of vectors only, it is also, when a *point* system is used, precisely fitted for bringing out polar-reciprocal relations. For if we have any equations expressing geometric relations in terms of points and lines in a plane region, or points, lines and planes in space, we have only in the first case to put lines for points and points for lines, and in the second to put

planes for points and points for planes, in order to obtain the polar reciprocal relations.

A few examples will now be presented to afford a comparison between this system and quaternions.

We will first give the equation by which Tait, in Art. 247 of his Quaternions, proves Pascal's Theorem together with the equation of precisely the same meaning in Grassmann's notation. They are respectively

$$S.V(V\alpha\beta\, V\delta\varepsilon)\, V(V\beta\gamma\, V\varepsilon\rho)\, V(V\gamma\delta\, V\rho\alpha) = 0$$

and

$$(\alpha\beta.\delta\varepsilon)(\beta\gamma.\varepsilon\rho)(\gamma\delta.\rho\alpha) = 0\,.$$

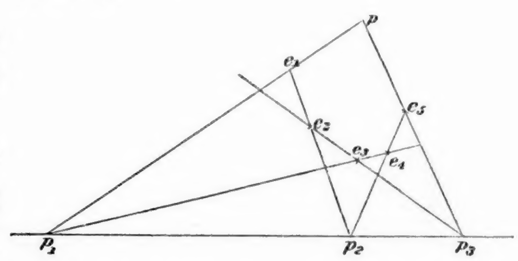
The comparative simplicity of the latter is apparent at a glance, and the ease of interpretation is as much greater as the labor of writing is less. There are ten capital letters in the first which are dispensed with in the last.

The equation is that of a cone on which lie each of the five vectors $\alpha, \beta, \gamma, \delta, \varepsilon$ drawn outwards from a common point, and is also the condition that the three vectors $\alpha\beta.\delta\varepsilon$, $\beta\gamma.\varepsilon\rho$, $\gamma\delta.\rho\alpha$ shall lie in one plane. Pascal's theorem is proved by considering a *section* of this cone.

If, however, we use *points* instead of vectors we have just what we want, the equation of a *conic* through five points, and the proof of the theorem follows immediately. If e_1, e_2, e_3, e_4, e_5 are five points, the conic passing through them may be written

$$(pe_1.e_3e_4)(e_1e_2.e_4e_5)(e_2e_3.e_5p) = 0\,,$$

which is also the condition that the three points in parentheses shall lie on one right line.



This equation possesses the further advantage that by merely substituting L's for p's and e's we have the equation

$$(LL_1.L_3L_4)(L_1L_2.L_4L_5)(L_2L_3.L_5L) = 0$$

which is the line equation of a conic tangent to five given lines, and gives at once Brianchon's theorem.

There are given two points e_0 and e_1, and two lines e_2e_4 and e_3e_5; two lines pass constantly through e_0 and e_1 respectively, and their points of intersection with e_2e_4 and e_3e_5 move along these lines with velocities bearing a constant ratio to each other: to find the locus of p the common point of the two moving lines.

BY QUATERNIONS.

Let $\overline{e_0e_2} = a\varepsilon_1 + b\varepsilon_2 = \varepsilon_4$, $\overline{e_2e_4} = \varepsilon_2$, $\overline{e_5e_3} = \varepsilon_3$, $\overline{e_0e_1} = c\varepsilon_1$, $\overline{e_1e_3} = f\varepsilon_1$ and $\varepsilon_3 = m\varepsilon_1 + n\varepsilon_2$: then $\rho = u(a\varepsilon_1 + b\varepsilon_2 + x\varepsilon_2)\ldots(\alpha)$ and $\rho = c\varepsilon_1 + v[f\varepsilon_1 + x(m\varepsilon_1 + n\varepsilon_2)]\ldots(\beta)$

Operate on (α) by $V\varepsilon_1$ and $V\varepsilon_2$ successively and divide thus

$$V\varepsilon_1\rho = u(b+x)\,V\varepsilon_1\varepsilon_2,$$
$$V\varepsilon_2\rho = -\,au\,V\varepsilon_1\varepsilon_2,$$

and $\dfrac{V\varepsilon_1\rho}{V\varepsilon_2\rho} = \dfrac{b+x}{-a}$: $\therefore\ x = \dfrac{V(a\varepsilon_1 + b\varepsilon_2)\rho}{-\,V\varepsilon_2\rho}$
$$= -\frac{V\varepsilon_4\rho}{V\varepsilon_2\rho}.$$

Similarly from (β)

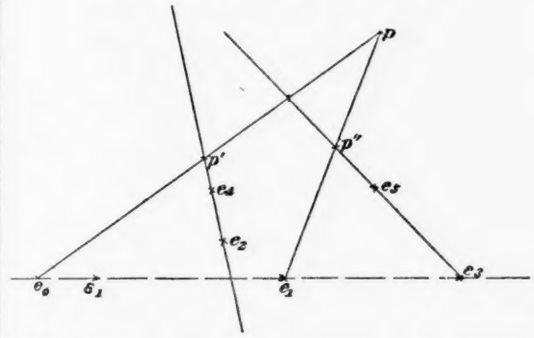
$$V\varepsilon_1\rho = vxn\,V\varepsilon_1\varepsilon_2,$$
$$V\varepsilon_2\rho = -\,c\,V\varepsilon_1\varepsilon_2 - v(f+mx)\,V\varepsilon_1\varepsilon_2,$$

$$\therefore\ \frac{V\varepsilon_1\rho}{V\varepsilon_2(\rho - c\varepsilon_1)} = \frac{-nx}{f+mx} = \frac{n\dfrac{V\varepsilon_4\rho}{V\varepsilon_2\rho}}{f - m\dfrac{V\varepsilon_4\rho}{V\varepsilon_2\rho}}$$

or $$\frac{V\varepsilon_1\rho}{V\varepsilon_2(\rho - c\varepsilon_1)} = \frac{n\,V\varepsilon_4\rho}{V(\varepsilon_2 f - m\varepsilon_4)\rho}$$

the equation of a conic because of the second degree in ρ.

BY CALCULUS OF POSITION.



$p' = e_2 + x(e_4 - e_2)$, $p'' = e_3 + x(e_5 - e_3)$ and also $e_0p'p = 0$, $e_1p''p = 0$.

Insert values of p' and p'',

$\therefore\ e_0e_2p + xe_0(e_4 - e_2)p = 0$, or $x = \dfrac{-\,e_0e_2p}{e_0(e_4 - e_2)p}$

and

$e_1e_3p + xe_1(e_5 - e_3)p = 0$, or $x = \dfrac{-\,e_1e_3p}{e_1(e_5 - e_3)p}$

$\therefore$ equating values of x

$$\frac{e_0e_2p}{e_0(e_4 - e_2)p} = \frac{e_1e_3p}{e_1(e_5 - e_3)p}$$

a conic because of the second degree in p.

From the complementary equation,

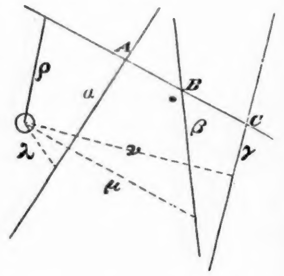
$$\frac{L_0L_2L}{L_0(L_4 - L_2)L} = \frac{L_1L_3L}{L_1(L_5 - L_3)L}$$

we may derive at once a polar reciprocal theorem.

Working with vectors but using Grassmann's system we should have saved the writing of twenty-seven unnecessary V's.

As one more example we will take the following. To find the equation of the surface generated by a right line moving on three right lines as directrices.

BY QUATERNIONS.



ABC is a generatrix of the surface, its equation being either

$$\rho = \lambda + x\alpha + u(\mu + y\beta - \lambda - x\alpha)\ldots(1)$$

or

$$\rho = \mu + y\beta + v(\nu + z\gamma - \mu - y\beta),\ldots(2)$$

from (1) we have

$$S\alpha\beta\rho = S\alpha\beta\lambda + u(S\alpha\beta\mu - S\alpha\beta\lambda);$$

$$\therefore \qquad u = \frac{S\alpha\beta(\rho-\lambda)}{S\alpha\beta(\mu-\lambda)},\ldots \qquad (3)$$

Similarly from (2)

$$v = \frac{S\beta\gamma(\rho-\mu)}{S\beta\gamma(\nu-\mu)}\ldots \qquad (4)$$

Operate on (1) by $S.V\gamma\alpha$

$$\therefore\ S\gamma\alpha\rho = S\gamma\alpha\lambda + uS\gamma\alpha(\mu + y\beta - \lambda)$$

$$\therefore\ y = \frac{S\gamma\alpha(\rho-\lambda) - uS\gamma\alpha(\mu-\lambda)}{uS\alpha\beta\gamma}\ [\text{and by (3)}]$$

$$y = \frac{S\gamma\alpha(\rho-\lambda)S\alpha\beta(\mu-\lambda) - S\gamma\alpha(\mu-\lambda)S\alpha\beta(\rho-\lambda)}{S\alpha\beta\gamma\, S\alpha\beta(\rho-\lambda)}$$

$$= \frac{S.V\gamma\alpha\, V\alpha\beta\, V(\mu-\lambda)(\rho-\lambda)}{S\alpha\beta\gamma S\alpha\beta(\rho-\lambda)}.$$

Operating on (2) by $S.V\gamma\alpha$ and substituting value of v we have by a similar reduction

$$y = \frac{S.V\beta\gamma\, V\gamma\alpha\, V(\nu-\mu)(\rho-\mu)}{S\alpha\beta\gamma S\beta\gamma(\rho-\nu)}.$$

Equating these values of y we have the equation of the surface of the second degree in ρ, viz.:

$$\frac{S.V\gamma\alpha\, V\alpha\beta\, V(\mu-\lambda)(\rho-\lambda)}{S\alpha\beta(\rho-\lambda)} = \frac{S.V\beta\gamma\, V\gamma\alpha\, V(\nu-\mu)(\rho-\mu)}{S\beta\gamma(\rho-\nu)}$$

BY CALCULUS OF POSITION.

Let the three lines be L_1, L_2, L_3. If through a point on L_2 planes be passed containing L_1 and L_3, then their common line will be a generatrix of the surface, for it will evidently cut L_1, L_2 and L_3. Let p be any point of this common line, then the two planes will be pL_1 and L_3p. These are to cut L_2 at the same point, the condition for which is

$$pL_1.L_2.L_3p = 0.$$

This being a scalar equation of the second degree in p represents a quadric, which is the required surface.

In working out the vector equation with the other system we should save the writing of *forty-six* unnecessary S's and V's. The last equation, for instance, would be

$$\frac{\gamma\alpha\,.\,\alpha\beta\,.(\mu-\lambda)(\rho-\lambda)}{\alpha\beta(\rho-\lambda)}=\frac{\beta\gamma\,.\,\gamma\alpha\,.(\nu-\mu)(\rho-\mu)}{\beta\gamma(\rho-\nu)}.$$

To illustrate the applicability of this method to Mechanics we will give a single example.

Let P_1, P_2, etc., be vectors representing in magnitude and direction any system of forces acting on a rigid body, and let e_1, e_2, etc., be points in their respective lines of action, then the forces are completely represented by e_1P_1, e_2P_2, etc.

The resultant action of the system is simply the sum of these forces, viz. $\sum\limits_1^n(eP)$: and for equilibrium the condition is

$$\sum_1^n(eP)=0.$$

Add and subtract $e_0\Sigma(P)$, e_0 being any point whatever;

$$\sum_1^n(eP)-e_0\sum_1^n(P)+e_0\sum_1^n(P)=0$$

or

$$e_0\sum_1^n(P)+\sum_1^n[(e-e_0)P]=0.$$

But one of these terms is a *point-vector*, and the other a *plane-vector;* therefore, being quantities of different kinds, in order to satisfy the equation they must be *separately* equal to zero.

$$\therefore\qquad \left.\begin{array}{l}\Sigma(P)=0\\ \Sigma[(e-e_0)P]=0\end{array}\right\}$$

The first signifies that the resultant *force* must be zero, the second that the resultant *couple* must be zero.

In closing I may remark that a principal reason for the slow introduction of Grassmann's method seems to be the great *generality* of his demonstrations as given in his books. They being usually given for space of the n^{th} order, the idea appears to have prevailed that the method is peculiarly adapted to *hyper-geometry*, which is actually the case, without, however, at all interfering with its special fitness for application to space of two or three dimensions.

On Quadruple Theta-Functions.

By Thomas Craig, *Johns Hopkins University.*

Part I.

The following paper is intended to be simply introductory to the theory of the theta-functions of four variables. I have followed the method adopted by Prof. Cayley in his well-known memoir on the double theta-functions; the earlier pages of the present Part being indeed simply the extension of the work already done by Prof. Cayley for the theta-functions with two arguments. In the present Part I give the elementary theorems connected with the quadruple theta-functions and a product table for the "square-products," *i. e.* the products of two functions having the same characteristics but not the same arguments. In Part II I intend to go more fully into the theory of the characteristics and to develop the relations existing between different groups of quadruple theta-functions, that is, the relations similar to the Göpel and Kummer relations connecting the products and squares of the double theta-functions. The only paper that I am aware of which directly treats of the quadruple functions is one by Nöther mentioned below. In this paper Nöther deals almost entirely with certain properties of the characteristics. It is known (the proof for completeness is given below) that every characteristic with the exception of $\binom{0000}{0000}$ can be divided in 128 different ways into the sum of two, viz. in 64 ways into the sum of an odd and of an even characteristic; in 36 different ways into the sum of two even characteristics, and in 28 different ways into the sum of two odd characteristics. Denoting any characteristic by (h) and in particular the characteristic $\binom{0000}{0000}$ by (0), the system of 28 pairs of odd characteristics in which each characteristic (h), excepting (0), can be divided is called a *group*, and this group Nöther calls a group-characteristic and denotes by $[\alpha]$. If

$$[\alpha] = (a_1) + (a_2) = (a'_1) + (a'_2) = \text{etc.}, \ldots$$

where (a_1), (a_2), (a'_1), (a'_2), etc., are all odd, Nöther says that (a_1), (a_2), (a'_1), etc., are *contained* in the group $[\alpha]$, and further that (a_1) and (a_2), etc., are *paired*—

giving thus in all 255 pairs. Nöther has here extended Weber's investigation on the triple theta-functions, particularly that part of the latter's work which refers to what he calls "*complete*" 7-*systems*; Nöther shows that there are for the quadruple functions certain analogous 8-systems, that is, systems of 8 odd characteristics for which the sum of any 5 is odd, and the sum of any 3 or of any 7 is even. In the last section of his paper he gives as an application of the relations existing between these theta-functions with zero arguments, the reduction to the least number of conditions which must hold in order that the theta-functions may become hyperelliptic functions, showing finally that this only requires the vanishing of *three* even theta-functions. In the following list of memoirs I have enumerated only those with which I am acquainted and know to have a direct bearing upon the subject of the quadruple functions. Although none but the one already mentioned deal directly with these functions, all refer to them more or less directly.

NÖTHER. Zur Theorie der Thetafunctionen von vier Argumenten: *Mathematische Annalen*, xiv, pp. 249–293.

—— Zur Theorie der Thetafunctionen von beliebig vielen Argumenten: *Mathematische Annalen*, xvi, pp. 270–344.

PRINGSHEIM. Zur Theorie der hyperelliptischen Functionen, insbesondere derjenigen dritter Ordnung ($\rho = 4$): *Mathematische Annalen*, xii, pp. 435–475.

STAHL. Das Additionstheorem der ϑ-Functionen mit p Argumenten: *Jour. für die reine und ange. Math.*, 88, pp. 117–130.

—— Beweis eines Satzes von *Riemann* über ϑ-characteristiken: *Jour. für die reine und ange. Math.*, 88, pp. 273–276.

FROBENIUS. Zur Theorie der Transformation der Thetafunctionen: *Jour. für die reine und ange. Math.*, 89, pp. 40–46.

—— Ueber das Additionstheorem der Thetafunctionen mehrerer Variabeln: *Jour. für die reine und ange. Math.*, 89, pp. 185–220.

BRIOSCHI. La relazione di Göpel per funzioni iperellittiche d'ordini qualunque: *Annali di Mat.*, Ser. II^a, Tomo X°, pp. 161–172.

WEIERSTRASS. Zur Theorie der Abel'schen Functionen: *Jour. für die reine und ange. Math.*, Vols. 47 and 52.

WEBER. Ueber die Transformationstheorie der Thetafunctionen: *Annali di Mat.*, Ser. II^a, Tomo IX°.

PRYM. Untersuchungen über die Riemann'sche Thetaformel und die Riemann'sche charakteristikentheorie: Leipzig, 1882.

Weber's paper I have never seen, but have come across numerous references to it; Prym's paper is published separately, and is a most valuable one for the student interested in the subject of theta-functions. Frobenius, in the second of his papers above mentioned, takes as his point of departure a certain result arrived at by Weber and subsequently specialized in a most important way by Hermite. Weber shows that the transformed theta-functions of ρ variables can be expressed as integral functions of the k^{th} order of the original $2^{2\rho}$ theta-functions, k denoting the order of the transformation. Hermite shows for the case $\rho=2$, that if ϑ_0, ϑ_1, ϑ_2, ϑ_3 denote four of the double ϑ-functions connected by the Göpel bi-quadratic relation, and if Θ_0, Θ_1, Θ_2, Θ_4 denote the transformed functions, the four functions Θ_a can be expressed as integral functions of the four corresponding functions ϑ_a. Frobenius discusses then the question as to whether or not a similar property can be shown to exist for the theta-functions of more than two arguments, and gives application of his results to the cases of $\rho=2$, 3 and 4. The subject of the transformation of the quadruple functions is reserved for the second part of the present paper. Before leaving this introductory section, however, reference should be made to two papers by Clifford, both contained in his "Mathematical Papers," viz., "On groups of periodic functions," which deals with multiple theta-functions and follows Rosenhain's method; and "Theory of marks of multiple theta-functions," in which Clifford partly follows the general methods of Riemann and partly the particular methods employed by Weber in his memoir "Theorie der Abelschen Functionen vom Geschlecht 3."

The theta-function of four variables depends upon 10 parameters,

$$(a_{11}a_{12}\ldots a_{44})$$

which are the coefficients of a quadric function of four ultimately disappearing integers, say m_1, m_2, m_3, m_4, viz.

$$(a_{11}a_{12}\ldots a_{44}\between m_1,\ldots m_4)^2;$$

upon four arguments u_1, u_2, u_3, u_4, and upon 8 characteristics α_i and β_i, $i=1, 2, 3, 4$, which are each either 0 or 1. The arrangement

$$\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}$$

is called a *characteristic*, or, in Clifford's notation, a *mark*. It is easy to see for α and β each equal 0 or 1 that we have in all $2^{2\times 4}=256$ different characteristics, and consequently 256 quadruple theta-functions.

The integers m_i will be assumed all even, and we write

$$\begin{pmatrix} m_1 m_2 m_3 m_4 \\ u_1\ u_2\ u_3\ u_4 \end{pmatrix} = \frac{1}{4}(a_{11}a_{12}\ldots a_{44}\between m_1 m_2 m_3 m_4)^2 + \frac{1}{2}\pi i(m_1u_1 + m_2u_2 + m_3u_3 + m_4u_4)$$

and equally

$$\begin{pmatrix} m_1+\alpha_1,\ m_2+\alpha_2,\ m_3+\alpha_3,\ m_4+\alpha_4 \\ u_1+\beta_1,\ u_2+\beta_2,\ u_3+\beta_3,\ u_4+\beta_4 \end{pmatrix} = \frac{1}{4}(a_{11}a_{12}\ldots a_{44}\between m_1+\alpha_1\ldots m_4+\alpha_4)^2$$

$$+\frac{1}{2}\pi i((m_1+\alpha_1)(u_1+\beta_1) + \ldots + (m_4+\alpha_4)(u_4+\beta_4)).$$

The quadruple theta-functions are now defined by the equation

$$\vartheta\begin{pmatrix} \alpha_1\alpha_2\alpha_3\alpha_4 \\ \beta_1\beta_2\beta_3\beta_4 \end{pmatrix}(u_1u_2u_3u_4) = \sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4} \exp.\begin{pmatrix} m_1+\alpha_1,\ m_2+\alpha_2,\ m_3+\alpha_3,\ m_4+\alpha_4 \\ u_1+\beta_1,\ u_2+\beta_2,\ u_3+\beta_3,\ u_4+\beta_4 \end{pmatrix}.$$

The summations extend over all positive and negative even integer values of $m_1 \ldots m_4$. For brevity this may be written simply as

$$\vartheta\begin{pmatrix} \alpha \\ \beta \end{pmatrix}(u) = \Sigma \exp.\begin{pmatrix} m+\alpha \\ u+\beta \end{pmatrix}.$$

It is well known that in order that the series here written may be convergent, the parameters a_{ik} must either be real, or if they are imaginary, that their real parts must be negative. The quantities α and β are all either zero or unity; it may happen that other values will appear for these quantities, but in each case, as will be seen, the odd values may be replaced by unity and the even values by zero. If $\Sigma\alpha\beta$ is even, the characteristic $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is said to be even; if $\Sigma\alpha\beta$ is odd, the characteristic is said to be odd. The number of odd and of even characteristics is readily computed, and it is as easily done for p-tuple as for quadruple functions. Let O_{p-1} and E_{p-1} denote the number of odd and of even characteristics for a $p-1$-tuple function; then clearly $O_{p-1} + E_{p-1} = 2^{2(p-1)}$. Prefix to each of the O_{p-1} odd characteristics $\begin{smallmatrix}0\\0\end{smallmatrix}, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}$ and to each of the E_{p-1} even characteristics $\begin{smallmatrix}1\\1\end{smallmatrix}$ and so get an odd characteristic for the p-tuple function. We have then

$$O_p = 3O_{p-1} + E_{p-1} = 2^{2(p-1)} + 2O_{p-1}$$
$$= 2^{2(p-1)} + 2.2^{2(p-2)} + 2^2.2^{2(p-3)} + \ldots + 2^{p-1} = 2^{p-1}(2^p - 1)$$

and this for $p = 4$ is $O_p = 120$. For the number of even marks we have

$$E_p = 2^{2p} - O_p = 2^{p-1}(2^p + 1)$$

or for $p = 4$, $E_p = 136$.

Given two characteristics $\binom{\alpha}{\beta}$ and $\binom{a}{b}$, their sum, or difference, is given by the symbol $\binom{\alpha \pm a}{\beta \pm b}$, *i. e.* for the case of 4 arguments

$$\begin{pmatrix}\alpha_1, \alpha_2, \alpha_3, \alpha_4\\ \beta_1, \beta_2, \beta_3, \beta_4\end{pmatrix} \pm \begin{pmatrix}a_1, a_2, a_3, a_4\\ b_1, b_2, b_3, b_4\end{pmatrix} = \begin{pmatrix}\alpha_1 \pm a_1, \alpha_2 \pm a_2, \alpha_3 \pm a_3, \alpha_4 \pm a_4\\ \beta_1 \pm b_1, \beta_2 \pm b_2, \beta_3 \pm b_3, \beta_4 \pm b_4\end{pmatrix}.$$

Denoting by $\binom{\alpha}{\beta}$ any characteristic, we may obviously write

$$\binom{\alpha}{\beta} = \binom{a}{b} + \binom{c}{d}$$

and either of the characteristics on the right-hand side of the equation may be arbitrarily assumed—giving, say for the other

$$\binom{c}{d} = \binom{\alpha}{\beta} - \binom{a}{b}.$$

Give now to $\binom{a}{b}$ all of the 2^{2p} possible values which it may have for a p-tuple theta-function, and we have a corresponding set of 2^{2p} values for $\binom{c}{d}$; but as it is quite clear that $$\binom{a}{b} + \binom{c}{d} = \binom{c}{d} + \binom{a}{b}$$
there are only $2^{2p} \div 2$ different divisors of $\binom{\alpha}{\beta}$, or, in other words, any characteristic $\binom{\alpha}{\beta}$ of a p-tuple theta-function can be divided in 2^{2p-1} different ways into the sum of two characteristics. For $p = 4$ we have as above stated 128 different ways of dividing the characteristic of a quadruple function into the sum of two characteristics. Each of the possible decompositions of a given characteristic $\binom{\alpha}{\beta}$ into the sum of two others $\binom{a}{b}$ and $\binom{c}{d}$ may take place in one of three different ways: first, both $\binom{a}{b}$ and $\binom{c}{d}$ may be even; second, both may be odd, and third, one may be even and the other odd. Denote the number of the cases where $\binom{a}{b}$ and $\binom{c}{d}$ are both even by ξ, the number when both are odd by η, and when one is even and the other odd by ζ, required to find the values of ξ, η and ζ for a p-tuple theta-function.

The following method of obtaining the values of these quantities is essentially the same as that given by Prym—which again is almost identical with Riemann's process. It is necessary first, however, to prove a subsidiary theorem. Suppose we have a given characteristic $\binom{\alpha}{\beta}$ *i. e.* $\begin{pmatrix}\alpha_1\alpha_2 \ldots \alpha_p\\ \beta_1\beta_2 \ldots \beta_p\end{pmatrix}$; this can be

divided into the sum of two others $\binom{a}{b}$ and $\binom{c}{d}$; assume either one of these, say $\binom{c}{d}$ as any one of the 2^{2p} characteristics of a p-tuple function. Write

$$H=\sum_{\binom{c}{d}}(-)^{\Sigma(a_i d_i+\beta_i c_i)}$$

in which i takes all values from 1 up to p, and where the larger Σ refers to all possible values of $\binom{c}{d}$. Each c and each d of course take only one of the other of the values 0 and 1. Suppose now either c_i or d_i to alter by unity, that is, let either or both of them become c_i+1 and d_i+1, then the order of the summation under the small Σ is changed, but the value of the entire sum is unaltered; on the other hand, however, if c_i is increased by unity, H alters by the factor $(-)^{\beta_i}$, and if d_i is altered by unity, H takes the factor $(-)^{a_i}$. Each of these factors being independent of $\binom{c}{d}$ can be placed before the large Σ. We have then

$$H=(-)^{a_i}H,\ H=(-)^{\beta_i}H,$$

that is $H=0$, whenever of any one of the $2p$ quantities a_i and β_i (all of which are of course either 0 or 1) one at least is $=1$; this excludes of course only the characteristic $\binom{0}{0}$. Now writing $\binom{a}{\beta}=\binom{a}{b}+\binom{c}{d}$, where $\binom{a}{\beta}$ is a given characteristic, write in place of $\binom{c}{d}$ all of the E_p even characteristics, and determine each corresponding $\binom{a}{b}$ by means of the equation $\binom{a}{b}=\binom{a}{\beta}-\binom{c}{d}$.

It is clear then that in the decomposition of the characteristic $\binom{a}{\beta}$ each of the ξ-divisions will come in twice and each of the ζ-divisions will come in once. Again give to $\binom{c}{d}$ all of the O_p odd values and we have that each of the η-divisions comes in twice and each of the ζ-divisions comes in once. Expressed algebraically these two statements are

$$E_p=2\xi+\zeta,\ O_p=2\eta+\zeta.$$

Form now the expression

$$(-)^{\Sigma c_i d_i}.(-)^{\Sigma a_i b_i}$$

and denote this by ϕ. It is obvious that ϕ takes the value $+1$ for either the ξ- or the η-divisions, and the value -1 for the ζ-divisions. Now since $\binom{a}{b}$ is known when $\binom{c}{d}$ is given, give to $\binom{c}{d}$ all its possible 2^{2p} values and take the

sum of the corresponding values of ϕ; call this sum σ; then since each possible division of $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ comes in twice, we have

$$\sigma = 2(\xi + \eta - \zeta).$$

It is quite clear that we can write

$$\phi = (-)^{\Sigma c_i d_i}.(-)^{\Sigma(\alpha_i + c_i)(\beta_i + d_i)} = (-)^{\Sigma \alpha_i \beta_i}.(-)^{\Sigma(\alpha_i d_i + \beta_i c_i)}.$$

Taking now the sum of all values of ϕ and we have, since as above shown

$$\sum_{\binom{c}{d}} (-)^{\Sigma(\alpha_i d_i + \beta_i c_i)} = 0$$

when the case $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$ is excluded,

$$\sigma = \sum_{\binom{c}{d}} \phi = (-)^{\Sigma \alpha_i \beta_i}.\sum_{\binom{c}{d}} (-)^{\Sigma(\alpha_i d_i + \beta_i c_i)} = 0,$$

or finally $$\xi + \eta - \zeta = 0.$$

Grouping together now the three equations obtained, viz.

$$E_p = 2\xi + \zeta \qquad O_p = 2\eta + \zeta \qquad 0 = \xi + \eta - \zeta$$

we have
$$3\xi + \eta = E_p = 3E_{p-1} + O_{p-1}$$
$$3\eta + \xi = O_p = 3O_{p-1} + E_{p-1}$$
$$\zeta = \xi + \eta$$

and finally
$$\xi = E_{p-1} = 2^{p-2}(2^{p-1} + 1)$$
$$\eta = O_{p-1} = 2^{p-2}(2^{p-1} - 1)$$
$$\zeta = E_{p-1} + O_{p-1} = 2^{2(p-1)}.$$

For $p = 4$ we have then for the number of divisions of a given characteristic $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ into the sum of two even characteristics $\begin{pmatrix}a\\ b\end{pmatrix}$ and $\begin{pmatrix}c\\ d\end{pmatrix}$ $\xi = 36$, into the sum of two odd $\eta = 28$, into the sum of one even and the other odd $\zeta = 64$, making in all $\xi + \eta + \zeta = 128$.

Leaving here for the present the consideration of the special properties of the characteristics, I now obtain some of the more elementary properties of the quadruple theta-functions. If each or any α be increased by an even integer there is no change in the value of the function. For suppose α to be replaced by $\alpha + 2x$ where x is an integer, then since m is even and takes all even values from $-\infty$ to $+\infty$ it is clear that $m + \alpha + 2x$ will take just the same values as $m + \alpha$ and so produce no effect on the function. If, however, the integers β be each or any of them increased by an even integer there is a change in the function. The quadratic term is obviously unaltered, but the linear term is easily seen to be increased by the quantity (y being an integer)

$$\pi i(m_1y_1+m_2y_2+m_3y_3+m_4y_4)+\pi i(\alpha_1y_1+\alpha_2y_2+\alpha_3y_3+\alpha_4y_4).$$

The first term of this is an even multiple of πi, say $=2n\pi i$; but $e^{2n\pi i}=1$, so that this term produces no effect. The second may be either an even or an odd multiple of πi, that is, may in the series give either the factor $+1$ or -1. We have then

$$\vartheta\binom{\alpha+2x}{\beta}(u)=\vartheta\binom{\alpha}{\beta}(u)$$

$$\vartheta\binom{\alpha}{\beta+2y}(u)=(-)^{\Sigma\beta y}\vartheta\binom{\alpha}{\beta}(u)$$

or combining these

$$\vartheta\binom{\alpha+2x}{\beta+2y}(u)=(-)^{\Sigma\beta y}\vartheta\binom{\alpha}{\beta}(u).$$

The only effect of altering the elements of the characteristic by odd integers is to interchange the functions. The even characteristics correspond to even theta-functions, and the odd characteristics to odd theta-functions, for remembering that m takes all (even) positive and negative values, it is obvious that $-m-\alpha$ takes precisely the same series of values as $m+\alpha$, so that in the function $\vartheta\binom{\alpha}{\beta}(-u)$ we may write the linear term as

$$\frac{1}{2}\pi i\Sigma(-m-\alpha)(-u+\beta)$$

or

$$\frac{1}{2}\pi i\Sigma(m+\alpha)(u+\beta)-\pi i\Sigma m\beta-\pi i\Sigma\alpha\beta.$$

Taking the exponential of this we have that the first term gives the linear part of the general term in the series; the second term gives the factor $+1$; the third term gives the factor $(-)^{\Sigma\alpha\beta}$, so that finally we have

$$\vartheta\binom{\alpha}{\beta}(-u)=(-)^{\Sigma\alpha\beta}\vartheta\binom{\alpha}{\beta}(u).$$

PERIODS.

Take an integer x, then we have obviously

$$\vartheta\binom{\alpha}{\beta}(u+x)=\vartheta\binom{\alpha}{\beta+x}(u).$$

so that when u is altered by an integer x the functions are interchanged. Now replace x by $2x$, *i. e.* alter u by an even integer, then we have

$$\vartheta\binom{\alpha}{\beta}(u+2x)=\vartheta\binom{\alpha}{\beta+2x}(u)=(-)^{\Sigma\alpha x}\vartheta\binom{\alpha}{\beta}(u)$$

and consequently the function is altered at most in its sign. Again replace $2x$ by $4x$ and we have from the last equation

$$\vartheta\binom{\alpha}{\beta}(u+4x)=\vartheta\binom{\alpha}{\beta}(u)$$

or there is no alteration.

Combining all these results we may say that the quadruple theta-functions have the quarter periods (1, 1, 1, 1), the half periods (2, 2, 2, 2) and the whole periods (4, 4, 4, 4).

Write $$(a_{11}a_{12}\ldots a_{44}\between x_1x_2x_3x_4)^2 = 2\Phi$$

and denote the derivative of this with respect to x_i by Φ_{x_i}, we have, of course, $a_{ik} = a_{ki}$ so that the quantities Φ_{x_i} have the values

$$\begin{aligned}
\Phi_{x_1} &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4\\
\Phi_{x_2} &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4\\
\Phi_{x_3} &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4\\
\Phi_{x_4} &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4
\end{aligned}$$

Now change u into $u + \frac{1}{\pi i}\,.\,\Phi_x$ and examine the function

$$\vartheta\binom{\alpha - x}{\beta}\left(u + \frac{1}{\pi i}\,.\,\Phi_x\right).$$

Writing for brevity $$(a_{11}a_{12}\ldots a_{44}) = (a) \text{ etc.,}$$

we have for the exponent in the general term the value

$$\tfrac{1}{4}(a\between m+\alpha)^2 + \tfrac{1}{2}\pi i\Sigma(m+\alpha)(u+\beta) + A$$

where A is given by the equation

$$\begin{aligned}
A = -\tfrac{1}{2}(a\between m+\alpha)\between x) &\quad + \tfrac{1}{2}\Sigma(m+\alpha)\Phi_x\\
+\tfrac{1}{4}(a\between x)^2 &\quad - \tfrac{1}{2}\pi i\Sigma x(u+\beta)\\
&\quad - \tfrac{1}{2}\Sigma x\Phi_x.
\end{aligned}$$

The terms in the right-hand columns are easily seen to be

$$\begin{aligned}
&= \tfrac{1}{2}(a\between m+\alpha\between x)\\
&-\tfrac{1}{2}\pi i\Sigma x(u+\beta)\\
&+\tfrac{1}{2}(a\between x)^2
\end{aligned}$$

so that we have $$A = -\tfrac{1}{4}(a\between x)^2 - \tfrac{1}{2}\pi i\Sigma x(u+\beta)$$

which is independent of m and consequently gives a factor for the entire series. We have then

$$\vartheta\binom{\alpha - x}{\beta}\left(u + \frac{1}{\pi i}\Phi_x\right) = e^A\,\vartheta\binom{\alpha}{\beta}(u)$$

or replacing (since x is an integer) α by $\alpha + x$

$$\vartheta\binom{\alpha}{\beta}\left(u + \frac{1}{\pi i}\Phi_x\right) = e^A\,\vartheta\binom{\alpha+x}{\beta}(u).$$

The change of u into $u + \frac{1}{\pi i}\Phi_x$ then interchanges the functions and affects each of them by the exponential factor e^A. The quantities $\frac{1}{\pi i}\Phi_x$ are called conjoint quarter quasi-periods.

The total number of the p-tuple theta functions is 2^{2p}, so for $p=4$ we have 256 quadruple functions, each of these has a particular for the characteristic

$$\binom{\alpha}{\beta}, = \binom{\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4}{\beta_1\,\beta_2\,\beta_3\,\beta_4}.$$

The following table gives the entire group of characteristics and suggests a notation for the functions; the upper line of the characteristic is invariable for each column and the lower line for each row.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0000 0000	1000 0000	0100 0000	0010 0000	0001 0000	1100 0000	0110 0000	0011 0000	1001 0000	1010 0000	0101 0000	1110 0000	0111 0000	1011 0000	1101 0000	1111 0000
2	0000 1000	1000 *	0100	0010	0001	1100 *	0110	0011	1001 *	1010 *	0101	1110 *	0111	1011 *	1101 *	1111 *
3	0000 0100	1000	0100 *	0010	0001	1100 *	0110 *	0011	1001	1010	0101 *	1110 *	0111 *	1011	1101 *	1111 *
4	0000 0010	1000	0100	0010 *	0001	1100	0110 *	0011 *	1001	1010 *	0101	1110 *	0111 *	1011 *	1101	1111 *
5	0000 0001	1000	0100	0010	0001 *	1100	0110	0011 *	1001 *	1010	0101 *	1110	0111 *	1011 *	1101 *	1111 *
6	0000 1100	1000 *	0100 *	0010	0001	1100	0110 *	0011	1001 *	1010 *	0101 *	1110	0111 *	1011 *	1101	1111
7	0000 0110	1000	0100 *	0010 *	0001	1100 *	0110	0011 *	1001	1010 *	0101 *	1110	0111	1011 *	1101 *	1111
8	0000 0011	1000	0100	0010 *	0001 *	1100	0110 *	0011	1001 *	1010 *	0101 *	1110 *	0111	1011	1101 *	1111
9	0000 1001	1000 *	0100	0010	0001 *	1100 *	0110	0011 *	1001	1010 *	0101 *	1110 *	0111 *	1011	1101	1111
10	0000 1010	1000 *	0100	0010 *	0001	1100 *	0110 *	0011 *	1001 *	1010	0101	1110	0111 *	1011	1101 *	1111
11	0000 0101	1000	0100 *	0010	0001 *	1100 *	0110 *	0011 *	1001 *	1010	0101	1110 *	0111	1011 *	1101	1111
12	0000 1110	1000 *	0100 *	0010 *	0001	1100	0110	0011 *	1001 *	1010	0101 *	1110 *	0111	1011	1101	1111 *
13	0000 0111	1000	0100 *	0010 *	0001 *	1100 *	0110	0011	1001 *	1010 *	0101	1110	0111 *	1011	1101	1111 *
14	0000 1011	1000 *	0100	0010 *	0001 *	1100 *	0110 *	0011	1001	1010	0101 *	1110	0111	1011 *	1101	1111 *
15	0000 1101	1000 *	0100 *	0010	0001 *	1100	0110 *	0011 *	1001	1010 *	0101	1110	0111	1011	1101 *	1111 *
16	0000 1111	1000 *	0100 *	0010 *	0001 *	1100	0110	0011	1001	1010	0101	1110 *	0111 *	1011 *	1101 *	1111

Where the lower line of the characteristic is left blank, it is understood to be merely a repetition of the lower line in the same row and first column. The asterisks indicate the odd functions, 120 in all, 8 in each row except the first and 8 in each column except the first. Each of the theta-functions may be denoted by ϑ_{ik} where i stands for the number of the row and k for the number of the column. In the following table, the odd functions alone are indicated. It will be readily seen that there is a perfect symmetry in the arrangement

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1																
2		*				*			*	*		*		*	*	*
3			*			*	*				*	*	*		*	*
4				*			*	*		*		*	*	*		*
5					*			*	*		*		*	*	*	*
6		*	*				*		*	*	*		*	*		
7			*	*		*		*		*	*			*	*	
8				*	*		*		*	*	*	*			*	
9		*			*	*		*		*	*	*	*			
10		*		*		*	*	*	*				*		*	
11			*		*	*	*	*	*			*		*		
12		*	*	*				*	*		*	*				*
13			*	*	*	*			*	*			*			*
14		*		*	*	*	*				*			*		*
15		*	*		*		*	*		*					*	*
16		*	*	*	*							*	*	*	*	

for corresponding to any odd function ϑ_{ik}, there is also a function ϑ_{ki}. In the following table I have simply arranged the functions so that there shall be an unbroken diagonal of the odd ones. This arrangement is no longer symmetrical, and it is not possible to make it so, since there are 120 odd functions, and as is easily seen it is only possible to put 15 in a diagonal, there must always be one more on one side of the diagonal than on the other. There are in this table 52 on the left-hand side and 53 on the right, which, together with the 15 in the diagonal, makes up the entire number 120. The numbers in half parenthesis denote the corresponding rows of the two preceding tables.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1																
2		*				*			*	*		*		*	*	*
3			*			*	*				*	*	*		*	*
4				*			*	*		*		*	*	*		*
5					*			*	*		*		*	*	*	*
7) 6			*	*		*		*		*	*			*	*	
6) 7		*	*				*		*	*	*		*	*		
9) 8		*			*	*		*		*	*	*	*			
8) 9				*	*		*		*	*	*	*			*	
13) 10			*	*	*	*			*	*			*			*
12) 11		*	*	*				*	*		*	*				*
11) 12			*		*	*	*	*	*			*		*		
10) 13		*		*		*	*	*	*				*		*	
14		*		*	*	*	*				*			*		*
15		*	*	*	*							*	*	*	*	
16		*	*		*		*	*		*					*	*

Continuing to follow Prof. Cayley, we next find an expression for the product of two of the quadruple functions, the characteristics of which are respectively $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ and $\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}$. As usual the two functions to be multiplied are taken as

$$\vartheta\begin{pmatrix} \alpha \\ \beta \end{pmatrix}(u + u') \text{ and } \vartheta\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix}(u - u')$$

these standing of course for

$$\vartheta\begin{pmatrix} \alpha_1\,\alpha_2\,\alpha_3\,\alpha_4 \\ \beta_1\,\beta_2\,\beta_3\,\beta_4 \end{pmatrix}(u_1 + u'_1,\ u_2 + u'_2,\ u_3 + u'_3,\ u_4 + u'_4)$$

and
$$\vartheta\begin{pmatrix} \alpha'_1\,\alpha'_2\,\alpha'_3\,\alpha'_4 \\ \beta'_1\,\beta'_2\,\beta'_3\,\beta'_4 \end{pmatrix}(u_1 - u'_1,\ u_2 - u'_2,\ u_3 - u'_3,\ u_4 - u'_4)$$

By the above symbolical method of writing the argument of the exponential, we have for the first of these

$$\begin{pmatrix} m + \alpha \\ u + u' + \beta \end{pmatrix} \text{ and for the second } \begin{pmatrix} m' + \alpha' \\ u - u' + \beta' \end{pmatrix}.$$

In the product we of course have as exponent the sum of these two quantities, viz., using all the previously indicated abbreviations, this is

$$\tfrac{1}{4}(a)(m+\alpha)^2+\tfrac{1}{2}\pi i\Sigma(m+\alpha)(u+u'+\beta)$$
$$+\tfrac{1}{4}(a)(m'+\alpha')^2+\tfrac{1}{2}\pi i\Sigma(m'+\alpha')(u-u'+\beta').$$

In order to obtain the exact form of the product, Prof. Cayley compares this sum with the sum of the two functions

$$\binom{\mu+\tfrac{1}{2}(\alpha+\alpha')}{2u+\beta+\beta'}\text{ and }\binom{\mu'+\tfrac{1}{2}(\alpha-\alpha')}{2u'+\beta-\beta'};$$

this is $$=\tfrac{1}{4}(2a)(\mu+\tfrac{1}{2}(\alpha+\alpha'))^2+\tfrac{1}{2}\pi i\Sigma[\mu+\tfrac{1}{2}(\alpha+\alpha')][2u+\beta+\beta']$$
$$+\tfrac{1}{4}(2a)(\mu'+\tfrac{1}{2}(\alpha-\alpha'))^2+\tfrac{1}{2}\pi i\Sigma[\mu'+\tfrac{1}{2}(\alpha-\alpha')][2u'+\beta-\beta'].$$

The two sums are made identical by writing

$$m_1+m'_1=2\mu_1,\ m_1-m'_1=2\mu'_1,$$
$$m_2+m'_2=2\mu_2,\ m_2-m'_2=2\mu'_2,$$
$$m_3+m'_3=2\mu_3,\ m_3-m'_3=2\mu'_3,$$
$$m_4+m'_4=2\mu_4,\ m_4-m'_4=2\mu'_4.$$

Subject then to the conditions implied in these relations, the product of the two quadruple functions is, when written out in full,

$$=\sum\text{exp.}\begin{pmatrix}\mu_1+\tfrac{1}{2}(\alpha_1+\alpha'_1), & \mu_2+\tfrac{1}{2}(\alpha_2+\alpha'_2), & \mu_3+\tfrac{1}{2}(\alpha_3+\alpha'_3), & \mu_4+\tfrac{1}{2}(\alpha_4+\alpha'_4)\\ 2u_1+\beta_1+\beta'_1), & 2u_2+\beta_2+\beta'_2, & 2u_3+\beta_3+\beta'_3, & 2u_4+\beta_4+\beta'_4\end{pmatrix}$$
$$\times\sum\text{exp.}\begin{pmatrix}\mu'_1+\tfrac{1}{2}(\alpha_1-\alpha'_1), & \mu'_2+\tfrac{1}{2}(\alpha_2-\alpha'_2), & \mu'_3+\tfrac{1}{2}(\alpha_3-\alpha'_3), & \mu'_4+\tfrac{1}{2}(\alpha_4-\alpha'_4)\\ 2u'_1+\beta_1-\beta'_1, & 2u'_2+\beta_2-\beta'_2, & 2u'_3+\beta_3-\beta'_3, & 2u'_4+\beta_4-\beta'_4\end{pmatrix}$$

The even integers m_i and m'_i may be said to be similar when they are both $\equiv 0$ or both $\equiv 2\,(\text{mod. }4)$, they may be said to be dissimilar when one of them is $\equiv 0$ and the other $\equiv 2\,(\text{mod. }4)$. There are then in all 16 cases, as shown in the following table. By a pair is meant $(m_i,\ m'_i)$ the same suffix for each letter.

	Number of Cases.
4 pairs similar, ...	1
3 pairs similar, 1 pair dissimilar, ...	4
2 pairs " 2 pairs "	6
1 pair " 3 pairs "	4
4 pairs dissimilar, ...	1
Total number of cases, ...	16

In the first of these cases the μ and μ' are even; in the second case we may write one pair odd and three pairs even (the pairs here referring to the Greek letters), etc. We have then finally for the product

$$\vartheta\binom{\alpha}{\beta}(u+u').\vartheta\binom{\alpha'}{\beta'}(u-u')=$$

$$\sum\Theta\left(\begin{matrix}\tfrac{1}{2}(\alpha+\alpha')+p\\ \beta+\beta'\end{matrix}\right)(2u).\Theta\left(\begin{matrix}\tfrac{1}{2}(\alpha-\alpha')+p\\ \beta-\beta'\end{matrix}\right)(2u');$$

here Θ is written instead of ϑ to denote that each a_{ik} is replaced by $2a_{ik}$ and the summation refers to the different values of p, given in the following table, viz.

p_1	p_2	p_3	p_4
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1
1	1	0	0
0	1	1	0
0	0	1	1
1	0	0	1
1	0	1	0
0	1	0	1
1	1	1	0
0	1	1	1
1	0	1	1
1	1	0	1
1	1	1	1

There are in all 136 even quadruple functions, these do not vanish when the arguments u_1, u_2, u_3 and u_4 are made $=0$, and consequently there are 136 constants corresponding to the zero values of the arguments. These constants may be denoted by c_{ik} and are given in the following table. Of course each c here written is understood to be c_{ik} where i and k denote respectively the number of the row and the number of the column. The asterisks are written instead of zeros to denote the zero values of the odd functions. It will be noticed here that both the asterisks and the c's are symmetrically arranged in the table. When u_1, u_2, u_3 and u_4 are each indefinitely small, each of the even functions is reduced to its zero value plus a quadric function of (u) and each of the odd functions are given by linear functions of (u). In the first of these cases we may write

$$\vartheta=c+(c'_{11},\ c'_{12},\ c'_{13},\ c'_{14},\ c'_{22},\ c'_{23},\ c'_{24},\ c'_{33},\ c'_{34},\ c'_{44})(u_1,\ u_2,\ u_3,\ u_4)^2$$

giving thus 1360 new constants c'_{ik}; in the second case we may write

$$\vartheta = (c''_1,\ c''_2,\ c''_3,\ c''_4 \between u_1,\ u_2,\ u_3,\ u_4)$$

giving 480 new constants c''_i.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c
2	c	*	c	c	c	*	c	c	*	*	c	*	c	*	*	
3	c	c	*	c	c	*	*	c	c	c	*	*	*	c	*	*
4	c	c	c	*	c	c	*	*	c	*	c	*	*	*	c	*
5	c	c	c	c	*	c	c	*	*	c	c	*	*	*	*	*
6	c	*	*	c	c	c	*	c	*	*	*	c	*	*	c	c
7	c	c	*	*	c	*	c	*	c	*	*	c	c	*	*	c
8	c	c	c	*	*	c	*	c	*	*	*	*	c	c	*	c
9	c	*	c	c	*	*	c	*	c	*	*	*	*	c	c	c
10	c	*	c	*	c	*	*	*	*	c	c	c	*	c	*	c
11	c	c	*	c	*	*	*	*	*	c	c	*	c	*	c	c
12	c	*	*	*	c	c	c	*	*	c	*	*	c	c	c	*
13	c	c	*	*	*	*	c	c	*	*	c	c	*	c	c	*
14	c	*	c	*	*	*	*	c	c	c	*	c	c	*	c	*
15	c	*	*	c	*	c	*	*	c	*	c	c	c	c	*	*
16	c	*	*	*	*	c	c	c	c	c	c	*	*	*	*	c

A complete product table for the quadruple theta-functions would be much too long to give, as it would contain 256×256 or 2^{16} products These would consist of what might be called a square-set of 256 products and 255 other sets each containing 256 products. By a square-set is meant a set consisting of products of the form $\vartheta_{ik}(u+u').\vartheta_{ik}(u-u')$; the remaining sets would have different suffixes for each ϑ. In the following table I give the products contained in the square-set, and instead of writing Θ with its proper suffix I write (following Prof. Cayley) X with a certain suffix; I give first the definition of the X's which refer to $2u_1$, $2u_2$, $2u_3$, $2u_4$; the accented X's refer of course to the Θ-functions with arguments $2u'_1$, $2u'_2$, $2u'_3$, $2u'_4$. In the remaining 255 sets it would be necessary to introduce some new symbols, inasmuch as the characteristics would not all be made up of 0 and 1.

Θ	0000	1000	0100	0010	0001	1100	0110	0011	1001	1010	0101	1110	0111	1011	1101	1111
0000																
1000																
0100																
0010																
0001																
1100																
0110																
0011																
1001																
1010																
0101																
1110																
0111																
1011																
1101																
1111																

=	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}
1																
2																
3																
4																
5																
6																
7																
8																
9																
10																
11																
12																
13																
14																
15																
16																

From this we have, for example

$$\vartheta\binom{0100}{1001} = X_{3,9},\ \vartheta\binom{1110}{0110} = X_{12,7},\ \vartheta\binom{1111}{1111} = X_{16,16},\ \text{etc.}$$

Instead of writing X_{ik} it will occasionally be a little more convenient to write X_k^i, the upper letter denoting the number of the row and the lower one the number of the column. In the following table giving the square-set products, I have written in the first column the suffixes belonging to the particular ϑ; at the top of the next 16 columns will be found the letter x; this is to be read as XX'; the last column contains the first suffix for both X and X, and the first *row* contains the second suffix belonging to X and X', viz. the figure on the left of the comma is the second suffix for X and the figure on the right of the comma is the second suffix for X'. The signs alone appear in every row after the first, the suffixes in the first row being the same, of course, for all of the sixteen equations in each sub-set. The table is divided into 16 different sub-sets; in the first sub-set the general term is $X_{ik}X'_{ik}$, *i. e.* the suffixes are the same for X and X'. This is not the case with the other fifteen sub-sets, but it will be noticed that if i be the number of the sub-set the first row and i^{th} column is always made up of *plus* signs. I give here a few examples illustrating the use of the tables.

$$\begin{aligned}
\vartheta_{1,1}\vartheta_{1,1}, = \vartheta\binom{0000}{0000}\vartheta\binom{0000}{0000}, = & X_{1,1}X'_{1,1} + X_{1,2}X_{1,2} + X_{1,3}X'_{1,3} + X_{1,4}X'_{1,4} \\
& + X_{1,5}X'_{1,5} + X_{1,6}X'_{1,6} + X_{1,7}X'_{1,7} + X_{1,8}X'_{1,8} \\
& + X_{1,9}X'_{1,9} + X_{1,10}X'_{1,10} + X_{1,11}X'_{1,11} + X_{1,12}X'_{1,12} \\
& + X_{1,13}X'_{1,13} + X_{1,14}X'_{1,14} + X_{1,15}X'_{1,15} + X_{1,16}X'_{1,16},
\end{aligned}$$

$$\begin{aligned}
\vartheta_{4,5}\vartheta_{4,5}, = \vartheta\binom{0001}{0010}\vartheta\binom{0001}{0010}, = & X_{1,5}X'_{1,1} + X_{1,9}X'_{1,2} + X_{1,11}X'_{1,3} - X_{1,8}X'_{1,4} \\
& + X_{1,1}X'_{1,5} + X_{1,15}X'_{1,6} - X_{1,13}X'_{1,7} - X_{1,4}X'_{1,8} \\
& + X_{1,2}X'_{1,9} - X_{1,14}X'_{1,10} + X_{1,3}X'_{1,11} - X_{1,16}X'_{1,12} \\
& - X_{1,7}X'_{1,13} - X_{1,10}X'_{1,14} + X_{1,6}X'_{1,15} - X_{1,12}X'_{1,16},
\end{aligned}$$

$$\begin{aligned}
\vartheta_{10,12}\vartheta_{10,12}, = \vartheta\binom{1110}{1010}\vartheta\binom{1110}{1010}, = & X_{1,12}X'_{1,1} - X_{1,7}X'_{1,2} + X_{1,10}X'_{1,3} - X_{1,6}X'_{1,4} \\
& + X_{1,16}X'_{1,5} - X_{1,4}X'_{1,6} - X_{1,2}X'_{1,7} - X_{1,15}X'_{1,8} \\
& - X_{1,13}X'_{1,9} + X_{1,3}X'_{1,10} + X_{1,14}X'_{1,11} + X_{1,1}X'_{1,12} \\
& - X_{1,9}X'_{1,13} + X_{1,11}X_{1,14} - X_{1,8}X'_{1,15} + X_{1,5}X'_{1,16}.
\end{aligned}$$

I

ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
1,1	+1	+2	+3	+4	+5	+6	+7	+8	+9	+10	+11	+12	+13	+14	+15	+16	1
2,1	+1	−2	+3	+4	+5	−6	+7	+8	−9	−10	+11	−12	+13	−14	−15	−16	1
3,1	+1	+2	−3	+4	+5	−6	−7	+8	+9	+10	−11	−12	−13	+14	−15	−16	1
4,1	+1	+2	+3	−4	+5	+6	−7	−8	+9	−10	+11	−12	−13	−14	+15	−16	1
5,1	+1	+2	+3	+4	−5	+6	+7	−8	−9	+10	−11	+12	−13	−14	−15	−16	1
6,1	+1	−2	−3	+4	+5	+6	−7	+8	−9	−10	−11	+12	−13	−14	+15	+16	1
7,1	+1	+2	−3	−4	+5	−6	+7	−8	+9	−10	−11	+12	+13	−14	−15	+16	1
8,1	+1	+2	+3	−4	−5	+6	−7	+8	−9	−10	−11	−12	+13	+14	−15	+16	1
9,1	+1	−2	+3	+4	−5	−6	+7	−8	+9	−10	−11	−12	−13	+14	+15	+16	1
10,1	+1	−2	+3	−4	+5	−6	−7	−8	−9	+10	+11	+12	−13	+14	−15	+16	1
11,1	+1	+2	−3	+4	−5	−6	−7	−8	−9	+10	+11	−12	+13	−14	+15	+16	1
12,1	+1	−2	−3	−4	+5	+6	+7	−8	−9	+10	−11	−12	+13	+14	+15	−16	1
13,1	+1	+2	−3	−4	−5	−6	+7	+8	−9	−10	+11	+12	−13	+14	+15	−16	1
14,1	+1	−2	+3	−4	−5	−6	−7	+8	+9	+10	−11	+12	+13	−14	+15	−16	1
15,1	+1	−2	−3	+4	−5	+6	−7	−8	+9	−10	+11	+12	+13	+14	−15	−16	1
16,1	+1	−2	−3	−4	−5	+6	+7	+8	+9	+10	+11	−12	−13	−14	−15	+16	1

II

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,2	+ 2,1	+ 1,2	+ 6,3	+ 10,4	+ 9,5	+ 3,6	+ 12,7	+ 14,8	+ 5,9	+ 4,10	+ 15,11	+ 7,12	+ 16,13	+ 8,14	+ 11,15	+ 13,16	1
*	2,2	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
	3,2	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
	4,2	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
	5,2	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
*	6,2	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7,2	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
	8,2	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
*	9,2	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
*	10,2	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,2	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
*	12,2	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
	13,2	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
*	14,2	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
*	15,2	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16,2	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

III

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,3	+ 3,1	+ 6,2	+ 1,3	+ 7,4	+ 11,5	+ 2,6	+ 4,7	+ 13,8	+ 15,9	+ 12,10	+ 5,11	+ 10,12	+ 8,13	+ 16,14	+ 9,15	+ 14,16	1
	2,3	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
*	3,3	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
	4,3	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
	5,3	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
*	6,3	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
*	7,3	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
	8,3	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
	9,3	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
	10,3	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
*	11,3	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,3	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
*	13,3	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
	14,3	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
*	15,3	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16.3	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

IV

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,4	+ 4,1	+ 10,2	+ 7,3	+ 1,4	+ 8,5	+ 12,6	+ 3.7	+ 5,8	+ 14,9	+ 2.10	+ 13,11	+ 6.12	+ 11.13	+ 9.14	+ 16.15	+ 15.16	1
	2,4	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
	3,4	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
*	4,4	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
	5,4	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
	6,4	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
*	7,4	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
*	8,4	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9,4	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10,4	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,4	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
*	12,4	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
*	13,4	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
*	14,4	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
	15,4	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
*	16,4	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

V

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,5	+ 5,1	+ 9,2	+ 11,3	+ 8,4	+ 1,5	+ 15,6	+ 13,7	+ 4,8	+ 2,9	+ 14,10	+ 3,11	+ 16,12	+ 7,13	+ 10,14	+ 6,15	+ 12,16	1
	2,5	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
	3,5	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
	4,5	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
*	5,5	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
	6,5	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
	7,5	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8,5	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
*	9,5	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
	10,5	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
*	11,5	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
	12,5	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
*	13,5	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
*	14,5	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
*	15,5	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16,5	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

VI

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,6	+ 6,1	+ 3,2	+ 2,3	+ 12,4	+ 15,5	+ 1,6	+ 10,7	+ 16,8	+ 11,9	+ 7,10	+ 9,11	+ 4,12	+ 14,13	+ 13,14	+ 5.15	+ 8,16	1
*	2,6	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
*	3,6	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
	4,6	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
	5,6	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
	6,6	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
*	7,6	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
	8,6	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
*	9,6	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
*	10,6	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
*	11,6	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
	12,6	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
*	13.6	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
*	14,6	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
	15,6	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
	16.6	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

VII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1.7	+ 7.1	+ 12,2	+ 4.3	+ 3.4	+ 13.5	+ 10,6	+ 1.7	+ 11,8	+ 16,9	+ 6,10	+ 8.11	+ 2.12	+ 5.13	+ 15,14	+ 14,15	+ 9,16	1
	2.7	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
*	3.7	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
*	4.7	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
	5.7	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
*	6.7	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7.7	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8.7	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9.7	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10.7	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
*	11.7	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
	12,7	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
	13.7	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
*	14.7	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
*	15.7	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
	16.7	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

VIII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,8	+ 8,1	+ 14,2	+ 13,3	+ 5,4	+ 4,5	+ 16,6	+ 11,7	+ 1.8	+ 10,9	+ 9.10	+ 7.11	+ 15,12	+ 3,13	+ 2,14	+ 12,15	+ 6,16	1
	2,8	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
	3,8	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
*	4,8	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
*	5,8	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
	6,8	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
*	7,8	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
	8,8	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
*	9,8	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
*	10,8	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
*	11,8	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,8	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
	13,8	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
	14,8	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
*	15,8	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
	16,8	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

IX

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,9	+ 9,1	+ 5,2	+ 15,3	+ 14,4	+ 2,5	+ 11,6	+ 16,7	+ 10,8	+ 1,9	+ 8,10	+ 6,11	+ 13,12	+ 12,13	+ 4,14	+ 3,15	+ 7,16	1
*	2,9	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
	3,9	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
	4,9	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
*	5,9	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
*	6,9	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7,9	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8,9	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9,9	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10,9	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
*	11,9	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,9	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
*	13,9	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
	14,9	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
	15,9	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
	16,9	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

X

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,10	+ 10,1	+ 4,2	+ 12,3	+ 2,4	+ 14,5	+ 7,6	+ 6,7	+ 9,8	+ 8,9	+ 1,10	+ 16,11	+ 3,12	+ 15,13	+ 5,14	+ 13,15	+ 11,16	1
*	2,10	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
	3,10	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
*	4,10	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
	5,10	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
*	6,10	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
*	7,10	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
*	8,10	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
*	9,10	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
	10,10	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
	11,10	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
	12,10	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
*	13,10	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
	14,10	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
*	15,10	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
	16,10	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

XI

ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
1,11	+ 11,1	+ 15,2	+ 5,3	+ 13,4	+ 3,5	+ 9,6	+ 8,7	+ 7,8	+ 6,9	+ 16,10	+ 1,11	+ 14,12	+ 4,13	+ 12,14	+ 2,15	+ 10,16	1
2,11	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
* 3,11	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
4,11	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
5,11	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
* 6,11	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
* 7,11	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
* 8,11	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
* 9,11	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
10,11	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
11,11	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
* 12,11	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
13,11	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
* 14,11	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
15,11	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
16,11	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

XII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,12	+ 12,1	+ 7,2	+ 10,3	+ 6,4	+ 16,5	+ 4,6	+ 2,7	+ 15,8	+ 13,9	+ 3,10	+ 14,11	+ 1,12	+ 9,13	+ 11,14	+ 8,15	+ 5,16	1
*	2,12	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
*	3,12	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
*	4,12	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
	5,12	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−	1
	6,12	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
	7,12	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
*	8,12	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
*	9,12	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
	10,12	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
*	11,12	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
*	12,12	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
	13,12	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
	14,12	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
	15,12	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
*	16,12	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XIII

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,13	+ 13,1	+ 16,2	+ 8,3	+ 11,4	+ 7,5	+ 14,6	+ 5,7	+ 3,8	+ 12,9	+ 15,10	+ 4,11	+ 9,12	+ 1,13	+ 6,14	+ 10,15	+ 2,16	1
	2,13	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−	1
*	3,13	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
*	4,13	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
*	5,13	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
*	6,13	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
	7,13	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
	8,13	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
*	9,13	−	+	−	−	+	+	−	+	−	+	+	+	+	−	−	−	1
*	10,13	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,13	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
	12,13	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
*	13,13	−	−	+	+	+	−	+	−	+	+	−	−	+	−	−	+	1
	14,13	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
	15,13	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
*	16,13	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XIV

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,14	+ 14,1	+ 8,2	+ 16,3	+ 9,4	+ 10,5	+ 13,6	+ 15,7	+ 2,8	+ 4,9	+ 5,10	+ 12,11	+ 11,12	+ 6,13	+ 1,14	+ 7,15	+ 3,16	1
*	2,14	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
	3,14	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−	1
*	4,14	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
*	5,14	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
*	6,14	−	+	+	−	−	−	+	−	+	+	+	−	+	+	−	−	1
*	7,14	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
	8,14	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
	9,14	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
	10,14	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
*	11,14	−	−	+	−	+	+	+	+	+	−	−	+	−	+	−	−	1
	12,14	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
	13,14	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
*	14,14	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
	15,14	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−	1
*	16.14	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XV

	ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
	1,15	+ 15,1	+ 11,2	+ 9,3	+ 16,4	+ 6,5	+ 5,6	+ 14,7	+ 12,8	+ 3,9	+ 13,10	+ 2,11	+ 8,12	+ 10,13	+ 7,14	+ 1,15	+ 4,16	1
*	2,15	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
*	3,15	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
	4,15	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−	1
*	5,15	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
	6,15	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
*	7,15	−	−	+	+	−	+	−	+	−	+	+	−	−	+	+	−	1
*	8,15	−	−	−	+	+	−	+	−	+	+	+	+	−	−	+	−	1
	9,15	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
*	10,15	−	+	−	+	−	+	+	+	+	−	−	−	+	−	+	−	1
	11,15	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
	12,15	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−	1
	13,15	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−	1
	14,15	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−	1
*	15,15	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
*	16,15	−	+	+	+	+	−	−	−	−	−	−	+	+	+	+	−	1

XVI

ϑ	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	Suff.
1,16	+ 16,1	+ 13,2	+ 14,3	+ 15,4	+ 12,5	+ 8,6	+ 9,7	+ 6,8	+ 7,9	+ 11,10	+ 10,11	+ 5,12	+ 2,13	+ 3,14	+ 4,15	+ 1,16	1
* 2,16	−	+	−	−	−	+	−	−	+	+	−	+	−	+	+	+	1
* 3,16	−	−	+	−	−	+	+	−	−	−	+	+	+	−	+	+	1
* 4,16	−	−	−	+	−	−	+	+	−	+	−	+	+	+	−	+	1
* 5,16	−	−	−	−	+	−	−	+	+	−	+	−	+	+	+	+	1
6,16	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+	1
7,16	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+	1
8,16	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+	1
9,16	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+	1
10,16	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+	1
11,16	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+	1
* 12,16	−	+	+	+	−	−	−	+	+	−	+	+	−	−	−	+	1
* 13,16	−	−	+	+	+	+	−	−	+	+	−	−	+	−	−	+	1
* 14,16	−	+	−	+	+	+	+	−	−	−	+	−	−	+	−	+	1
* 15,16	−	+	+	−	+	−	+	+	−	+	−	−	−	−	+	+	1
16.16	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	1

There is a certain symmetry in these tables, although it is hardly apparent in this (the most convenient) way of writing them. In the first table, the signs are all the same in the first row and in the first column; leaving out these lines and taking the fifteen by fifteen square which is left, it will be noticed that the signs are arranged in the same way in both the first row and the first column; forming again the fourteen by fourteen square, etc., the same thing will be noticed to hold true. I do not see any way of arranging the next fifteen tables in order to bring them into this form.

By making the quantities u'_1, u'_2, u'_3 and u'_4 all equal to zero, we have a linear function of the X_1, X_2, X_3, etc., equal to the square of a certain ϑ; and further making the u_1, u_2, u_3 and u_4 vanish, we have the constants c_{ik} expressed as quadratic functions of the zero values of X_{ik} and X'_{ik} (the zero value of X_{ik} being of course equal to the zero value of X'_{ik}). The more general form for the products is indicated by Nöther in his memoir above referred to, where he shows that the product

$$\vartheta(u+v+w).\vartheta(u-v)$$

can be expressed linearly and homogeneously by sixteen products of the form

$$\vartheta_a(u+w).\vartheta_a(u).$$

This he obtains from a general theorem in the theory of the ϑ-functions of any number of arguments, viz. (for four arguments) "*that between any* $2^4+1=17$ *products of the form*

$$\vartheta_a(u+v')\vartheta_a(u+v'')$$

which have equal arguments u_1, u_2, u_3, u_4, *and equal sums*

$$v'_1+v''_1,\ v'_2+v''_2,\ v'_3+v''_3,\ v'_4+v''_4,$$

there exists a linear homogeneous relation, the coefficients of which are independent of u." The sums are in the present case all equal zero, as $v''=-v',=u'$.

The zero values of X_i and X_i' being the same, they may be represented by a_i. In the following tables are arranged the squares of the zero values of the theta functions which are denoted by c_{ik}, only the even functions are written down, since the odd functions are zero for zero values of the arguments. The first table containing 16 rows corresponds to table I above; the second, third, etc., tables containing only 8 rows each, correspond to tables II, III, etc., above. The first column contains the suffixes of each c and the following columns contain the suffixes of the a's. The sign of each term is written above it just

as in the preceding tables, of course the arrangement of signs is the same as in these tables. When two suffixes are separated by a comma, the product of the corresponding α's is meant; thus $\boxed{\overset{+}{i,k}}$ is to be read $+\alpha_i.\alpha_k$. Where there is only one suffix written down, the square of the corresponding α is meant; thus $\boxed{\overset{+}{i}}$ is to be read $+\alpha_i^2$. These remarks do not of course apply to the first column, as this contains the regular suffixes of each ϑ as before defined, the comma is there written simply for convenience in reading. Another set of 16 tables might be given here giving the squared values of each ϑ_{ik} in terms of X_i and α_i, viz. as already mentioned linear functions of the X_i. This is not necessary, however, as these relations can be obtained at once from the last 16 tables by simply reading for x, αX instead of XX'. Another point is to be noted in using these tables, viz. that $\boxed{\overset{\pm}{i,k}}$ has the same value as $\boxed{\overset{\pm}{k,i}}$. It will be observed then in every table after the first that there are only eight terms, each product $\alpha_i.\alpha_k$ coming in twice and each product $\alpha_i.\alpha_k$ having the same sign as $\alpha_k.\alpha_i$. The odd functions in the previous set of tables are marked by an asterisk at the left-hand side of the characteristics contained in the first column; for these functions it will be observed that to each $\pm\alpha_i.\alpha_k$ there corresponds $\mp\alpha_i.\alpha_k$ in another column—the odd values, therefore, vanishing for zero values of the arguments, or in the general case (referring of course only to the odd functions) there is corresponding to any term of the form $\pm X_i, X_k'$ another term of the form $\mp X_k, X_i'$. Table I contains only sums and differences of *squares* of α_i, and the other tables contain only products of the form $2\alpha_i.\alpha_k$. This arrangement differs slightly from that given by Prof. Cayley for the double theta-functions. The construction of tables giving the values of $c_i.c_k$ would obviously involve a great deal of work, and from the difficulty in using them would be practically of no value.

1

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\overset{+}{8}$	$\overset{+}{9}$	$\overset{+}{10}$	$\overset{+}{11}$	$\overset{+}{12}$	$\overset{+}{13}$	$\overset{+}{14}$	$\overset{+}{15}$	$\overset{+}{16}$
2,1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{-}{6}$	$\overset{+}{7}$	$\overset{+}{8}$	$\overset{-}{9}$	$\overset{-}{10}$	$\overset{+}{11}$	$\overset{-}{12}$	$\overset{+}{13}$	$\overset{-}{14}$	$\overset{-}{15}$	$\overset{-}{16}$
3,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{-}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{-}{6}$	$\overset{-}{7}$	$\overset{+}{8}$	$\overset{+}{9}$	$\overset{+}{10}$	$\overset{-}{11}$	$\overset{-}{12}$	$\overset{-}{13}$	$\overset{+}{14}$	$\overset{-}{15}$	$\overset{-}{16}$
4,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{-}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{-}{7}$	$\overset{-}{8}$	$\overset{+}{9}$	$\overset{-}{10}$	$\overset{+}{11}$	$\overset{-}{12}$	$\overset{-}{13}$	$\overset{-}{14}$	$\overset{+}{15}$	$\overset{-}{16}$
5,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{-}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\overset{-}{8}$	$\overset{-}{9}$	$\overset{+}{10}$	$\overset{-}{11}$	$\overset{+}{12}$	$\overset{-}{13}$	$\overset{-}{14}$	$\overset{-}{15}$	$\overset{-}{16}$
6,1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{-}{3}$	$\overset{+}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{-}{7}$	$\overset{+}{8}$	$\overset{-}{9}$	$\overset{-}{10}$	$\overset{-}{11}$	$\overset{+}{12}$	$\overset{-}{13}$	$\overset{-}{14}$	$\overset{+}{15}$	$\overset{+}{16}$
7,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{-}{3}$	$\overset{-}{4}$	$\overset{+}{5}$	$\overset{-}{6}$	$\overset{+}{7}$	$\overset{-}{8}$	$\overset{+}{9}$	$\overset{-}{10}$	$\overset{-}{11}$	$\overset{+}{12}$	$\overset{+}{13}$	$\overset{-}{14}$	$\overset{-}{15}$	$\overset{+}{16}$
8,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{+}{3}$	$\overset{-}{4}$	$\overset{-}{5}$	$\overset{+}{6}$	$\overset{-}{7}$	$\overset{+}{8}$	$\overset{-}{9}$	$\overset{-}{10}$	$\overset{-}{11}$	$\overset{-}{12}$	$\overset{+}{13}$	$\overset{+}{14}$	$\overset{-}{15}$	$\overset{+}{16}$
9.1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{+}{3}$	$\overset{+}{4}$	$\overset{-}{5}$	$\overset{-}{6}$	$\overset{+}{7}$	$\overset{-}{8}$	$\overset{+}{9}$	$\overset{-}{10}$	$\overset{-}{11}$	$\overset{-}{12}$	$\overset{-}{13}$	$\overset{+}{14}$	$\overset{+}{15}$	$\overset{+}{16}$
10,1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{+}{3}$	$\overset{-}{4}$	$\overset{+}{5}$	$\overset{-}{6}$	$\overset{-}{7}$	$\overset{-}{8}$	$\overset{-}{9}$	$\overset{+}{10}$	$\overset{+}{11}$	$\overset{+}{12}$	$\overset{-}{13}$	$\overset{+}{14}$	$\overset{-}{15}$	$\overset{+}{16}$
11,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{-}{3}$	$\overset{+}{4}$	$\overset{-}{5}$	$\overset{-}{6}$	$\overset{-}{7}$	$\overset{-}{8}$	$\overset{-}{9}$	$\overset{+}{10}$	$\overset{+}{11}$	$\overset{-}{12}$	$\overset{+}{13}$	$\overset{-}{14}$	$\overset{+}{15}$	$\overset{+}{16}$
12,1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{-}{3}$	$\overset{-}{4}$	$\overset{+}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\overset{-}{8}$	$\overset{-}{9}$	$\overset{+}{10}$	$\overset{-}{11}$	$\overset{-}{12}$	$\overset{+}{13}$	$\overset{+}{14}$	$\overset{+}{15}$	$\overset{-}{16}$
13,1	$\overset{+}{1}$	$\overset{+}{2}$	$\overset{-}{3}$	$\overset{-}{4}$	$\overset{-}{5}$	$\overset{-}{6}$	$\overset{+}{7}$	$\overset{+}{8}$	$\overset{-}{9}$	$\overset{-}{10}$	$\overset{+}{11}$	$\overset{+}{12}$	$\overset{-}{13}$	$\overset{+}{14}$	$\overset{+}{15}$	$\overset{-}{16}$
14,1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{+}{3}$	$\overset{-}{4}$	$\overset{-}{5}$	$\overset{-}{6}$	$\overset{-}{7}$	$\overset{+}{8}$	$\overset{+}{9}$	$\overset{+}{10}$	$\overset{-}{11}$	$\overset{+}{12}$	$\overset{+}{13}$	$\overset{-}{14}$	$\overset{+}{15}$	$\overset{-}{16}$
15,1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{-}{3}$	$\overset{+}{4}$	$\overset{-}{5}$	$\overset{+}{6}$	$\overset{-}{7}$	$\overset{-}{8}$	$\overset{+}{9}$	$\overset{-}{10}$	$\overset{+}{11}$	$\overset{+}{12}$	$\overset{+}{13}$	$\overset{+}{14}$	$\overset{-}{15}$	$\overset{-}{16}$
16,1	$\overset{+}{1}$	$\overset{-}{2}$	$\overset{-}{3}$	$\overset{-}{4}$	$\overset{-}{5}$	$\overset{+}{6}$	$\overset{+}{7}$	$\overset{+}{8}$	$\overset{+}{9}$	$\overset{+}{10}$	$\overset{+}{11}$	$\overset{-}{12}$	$\overset{-}{13}$	$\overset{-}{14}$	$\overset{-}{15}$	$\overset{+}{16}$

2

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,2	+ 2,1	+ 1,2	+ 6,3	+ 10,4	+ 9,5	+ 3,6	+ 12,7	+ 14,8	+ 5.9	+ 4,10	+ 15,11	+ 7,12	+ 16,13	+ 8,14	+ 11,15	+ 13,16
3,2	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
4,2	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
5,2	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
7,2	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
8,2	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
11,2	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
13,2	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−

3

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,3	+ 3,1	+ 6,2	+ 1,3	+ 7,4	+ 11,5	+ 2,6	+ 4,7	+ 13,8	+ 15,9	+ 12.10	+ 5,11	+ 10,12	+ 8.13	+ 16,14	+ 9,15	+ 14,16
2,3	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
4,3	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
5,3	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
8,3	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
9,3	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
10,3	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
14,3	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−

4

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,4	+ 4,1	+ 10,2	+ 7,3	+ 1,4	+ 8,5	+ 12,6	+ 3,7	+ 5,8	+ 14,9	+ 2,10	+ 13,11	+ 6,12	+ 11,13	+ 9,14	+ 16,15	+ 15,16
2,4	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
3,4	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
5,4	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
6,4	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
9,4	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
11,4	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
15,4	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

5

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,5	+ 5,1	+ 9,2	+ 11,3	+ 8,4	+ 1,5	+ 15,6	+ 13,7	+ 4,8	+ 2,9	+ 14,10	+ 3,11	+ 16,12	+ 7,13	+ 10,14	+ 6,15	+ 12,16
2,5	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
3,5	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
4,5	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
6,5	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
7,5	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
10,5	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
12,5	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−

6

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,6	+ 6,1	+ 3,2	+ 2,3	+ 12,4	+ 15,5	+ 1,6	+ 10,7	+ 16,8	+ 11,9	+ 7,10	+ 9,11	+ 4,12	+ 14,13	+ 13,14	+ 5,15	+ 8,16
4,6	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
5,6	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
6,6	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
8,6	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
12,6	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
15,6	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−
16,6	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

7

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,7	+ 7,1	+ 12,2	+ 4,3	+ 3,4	+ 13,5	+ 10,6	+ 1,7	+ 11,8	+ 16,9	+ 6,10	+ 8,11	+ 2,12	+ 5,13	+ 15,14	+ 14,15	+ 9,16
2,7	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
5,7	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
7,7	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
9,7	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
12,7	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
13,7	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
16,7	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

8

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,8	+ 8.1	+ 14,2	+ 13,3	+ 5,4	+ 4,5	+ 16,6	+ 11,7	+ 1,8	+ 10,9	+ 9.10	+ 7.11	+ 15,12	+ 3,13	+ 2,14	+ 12,15	+ 6,16
2,8	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
3,8	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
6.8	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
8,8	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
13.8	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
14,8	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
16.8	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

9

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,9	+ 9,1	+ 5,2	+ 15,3	+ 14,4	+ 2,5	+ 11,6	+ 16,7	+ 10,8	+ 1,9	+ 8,10	+ 6,11	+ 13,12	+ 12,13	+ 4,14	+ 3,15	+ 7,16
3,9	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
4.9	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
7.9	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
9,9	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
14.9	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
15,9	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−
16,9	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

10

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,10	+ 10,1	+ 4,2	+ 12,3	+ 2,4	+ 14,5	+ 7,6	+ 6,7	+ 9,8	+ 8,9	+ 1,10	+ 16,11	+ 3,12	+ 15,13	+ 5,14	+ 13,15	+ 11,16
3,10	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
5,10	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
10,10	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
11,10	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
12,10	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
14,10	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
16,10	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

11

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,11	+ 11,1	+ 15,2	+ 5,3	+ 13,4	+ 3,5	+ 9,6	+ 8,7	+ 7,8	+ 6,9	+ 16,10	+ 1,11	+ 14,12	+ 4,13	+ 12,14	+ 2,15	+ 10,16
2,11	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
4,11	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
10,11	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
11,11	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
13,11	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
15,11	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−
16,11	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

12

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,12	+ 12,1	+ 7,2	+ 10,3	+ 6,4	+ 16,5	+ 4,6	+ 2,7	+ 15,8	+ 13,9	+ 3,10	+ 14,11	+ 1,12	+ 9,13	+ 11,14	+ 8,15	+ 5,16
5,12	+	+	+	+	−	+	+	−	−	+	−	+	−	−	−	−
6,12	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
7,12	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
10,12	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
13,12	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
14,12	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
15,12	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

13

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,13	+ 13,1	+ 16,2	+ 8,3	+ 11,4	+ 7,5	+ 14,6	+ 5,7	+ 3,8	+ 12,9	+ 15,10	+ 4,11	+ 9,12	+ 1,13	+ 6,14	+ 10,15	+ 2,16
2,13	+	−	+	+	+	−	+	+	−	−	+	−	+	−	−	−
7,13	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
8,13	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
11,13	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
12.13	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
14,13	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−
15,13	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

14

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,14	+ 14,1	+ 8,2	+ 16,3	+ 9,4	+ 10,5	+ 13,6	+ 15,7	+ 2,8	+ 4,9	+ 5,10	+ 12,11	+ 11,12	+ 6,13	+ 1,14	+ 7,15	+ 3,16
3,14	+	+	−	+	+	−	−	+	+	+	−	−	−	+	−	−
8,14	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
9,14	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
10,14	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
12,14	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
13,14	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
15,14	+	−	−	+	−	+	−	−	+	−	+	+	+	+	−	−

15

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,15	+ 15,1	+ 11,2	+ 9,3	+ 16,4	+ 6,5	+ 5,6	+ 14,7	+ 12,8	+ 3,9	+ 13,10	+ 2,11	+ 8,12	+ 10,13	+ 7,14	+ 1,15	+ 4,16
4,15	+	+	+	−	+	+	−	−	+	−	+	−	−	−	+	−
6,15	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
9,15	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
11,15	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
12,15	+	−	−	−	+	+	+	−	−	+	−	−	+	+	+	−
13,15	+	+	−	−	−	−	+	+	−	−	+	+	−	+	+	−
14,15	+	−	+	−	−	−	−	+	+	+	−	+	+	−	+	−

16

c^2	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1,16	+ 16,1	+ 13,2	+ 14,3	+ 15,4	+ 12,5	+ 8,6	+ 9,7	+ 6,8	+ 7,9	+ 11,10	+ 10,11	+ 5,12	+ 2,13	+ 3,14	+ 4,15	+ 1,16
6,16	+	−	−	+	+	+	−	+	−	−	−	+	−	−	+	+
7,16	+	+	−	−	+	−	+	−	+	−	−	+	+	−	−	+
8,16	+	+	+	−	−	+	−	+	−	−	−	−	+	+	−	+
9,16	+	−	+	+	−	−	+	−	+	−	−	−	−	+	+	+
10,16	+	−	+	−	+	−	−	−	−	+	+	+	−	+	−	+
11,16	+	+	−	+	−	−	−	−	−	+	+	−	+	−	+	+
16,16	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+

The following examples will serve to illustrate the method of using the tables.

TABLE 1.

$$c^2_{1,1}=\alpha_1^2+\alpha_2^2+\alpha_3^2+\alpha_4^2+\alpha_5^2+\alpha_6^2+\alpha_7^2+\alpha_8^2+\alpha_9^2+\alpha_{10}^2+\alpha_{11}^2+\alpha_{12}^2+\alpha_{13}^2+\alpha_{14}^2+\alpha_{15}^2+\alpha_{16}^2$$
$$c^2_{4,1}=\alpha_1^2+\alpha_2^2+\alpha_3^2-\alpha_4^2+\alpha_5^2+\alpha_6^2-\alpha_7^2-\alpha_8^2+\alpha_9^2-\alpha_{10}^2+\alpha_{11}^2-\alpha_{12}^2-\alpha_{13}^2-\alpha_{14}^2+\alpha_{15}^2-\alpha_{16}^2$$
$$c^2_{9,1}=\alpha_1^2-\alpha_2^2+\alpha_3^2+\alpha_4^2-\alpha_5^2-\alpha_6^2+\alpha_7^2-\alpha_8^2+\alpha_9^2-\alpha_{10}^2-\alpha_{11}^2-\alpha_{12}^2-\alpha_{13}^2+\alpha_{14}^2+\alpha_{15}^2+\alpha_{16}^2$$
$$c^2_{16,1}=\alpha_1^2-\alpha_2^2-\alpha_3^2-\alpha_4^2-\alpha_5^2+\alpha_6^2+\alpha_7^2+\alpha_8^2+\alpha_9^2+\alpha_{10}^2+\alpha_{11}^2-\alpha_{12}^2-\alpha_{13}^2-\alpha_{14}^2-\alpha_{15}^2+\alpha_{16}^2.$$

TABLE 4.

$$c^2_{1,4}=2(\alpha_4\alpha_1+\alpha_{10}\alpha_2+\alpha_7\alpha_3+\alpha_1\alpha_4+\alpha_8\alpha_5+\alpha_{12}\alpha_6+\alpha_{14}\alpha_9+\alpha_{13}\alpha_{11})$$
$$c^2_{11,4}=2(\alpha_4\alpha_1+\alpha_{10}\alpha_2-\alpha_7\alpha_3+\alpha_1\alpha_4-\alpha_8\alpha_5-\alpha_{12}\alpha_6-\alpha_{14}\alpha_9+\alpha_{13}\alpha_{11}).$$

TABLE 10.

$$c^2_{5,10}=2(\alpha_{10}\alpha_1+\alpha_4\alpha_2+\alpha_{12}\alpha_3-\alpha_{14}\alpha_5+\alpha_7\alpha_6-\alpha_9\alpha_8-\alpha_{16}\alpha_{11}-\alpha_{15}\alpha_{13})$$
$$c^2_{16,10}=2(\alpha_{10}\alpha_1-\alpha_4\alpha_2-\alpha_{12}\alpha_3-\alpha_{14}\alpha_5+\alpha_7\alpha_6+\alpha_9\alpha_8+\alpha_{16}\alpha_{11}-\alpha_{15}\alpha_{13}).$$

TABLE 15.

$$c^2_{6,15}=2(\alpha_{15}\alpha_1-\alpha_{11}\alpha_2-\alpha_9\alpha_3+\alpha_{16}\alpha_4+\alpha_6\alpha_5-\alpha_{14}\alpha_7+\alpha_{12}\alpha_8-\alpha_{13}\alpha_{10})$$
$$c^2_{12,15}=2(\alpha_{15}\alpha_1-\alpha_{11}\alpha_2-\alpha_9\alpha_3-\alpha_{16}\alpha_4+\alpha_6\alpha_5+\alpha_{14}\alpha_7-\alpha_{12}\alpha_8+\alpha_{13}\alpha_{10}).$$

The relations connecting these zero values would be determined with considerable difficulty from the above tables—the difficulty lying in the fact of the great number of terms in the α's which go to make up a given c. The following relation is obtained, however, without much trouble, viz.

$$\begin{vmatrix} c^2_{1,1}, & c^2_{2,1}, & c^2_{3,1}, & c^2_{4,1} \\ c^2_{5,1}, & c^2_{6,1}, & c^2_{7,1}, & c^2_{8,1} \\ c^2_{9,1}, & c^2_{10,1}, & c^2_{11,1}, & c^2_{12,1} \\ c^2_{13,1}, & c^2_{14,1}, & c^2_{15,1}, & c^2_{16,1} \end{vmatrix} = 16 \begin{vmatrix} \alpha^2_{16}, & \alpha^2_{15}, & \alpha^2_{14}, & \alpha^2_{13} \\ \alpha^2_{12}, & \alpha^2_{11}, & \alpha^2_{10}, & \alpha^2_{9} \\ \alpha^2_{8}, & \alpha^2_{7}, & \alpha^2_{6}, & \alpha^2_{5} \\ \alpha^2_{4}, & \alpha^2_{3}, & \alpha^2_{2}, & \alpha^2_{1} \end{vmatrix}$$

corresponding to the relation, in Prof. Cayley's notation for the double theta-functions

$$\begin{vmatrix} c^2_0, & c^2_4 \\ c^2_8, & c^2_{12} \end{vmatrix} = \begin{vmatrix} \delta^2, & \gamma^2 \\ \beta^2, & \alpha^2 \end{vmatrix}, \; c^2_{15} c^2_3.$$

The suffixes in this notation should each be increased by unity to make them correspond to the notation which I have used, and the $\alpha, \beta, \gamma, \delta$ should be replaced by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$.

Brioschi, in his paper above referred to, has given some general formulas connected with the extension of the Göpel biquadratic relations between double theta-functions to the case of functions of n arguments. These relations, for the present case of four arguments, could, of course, be directly obtained from the tables given above, together with the remaining set of product tables, which would obviously require too much space to write down. I only mention this paper of Brioschi's which, though brief and in some respects unsatisfactory, is the only one I know of in which an attempt is made to generalize the Göpel relations to the case of any number of arguments. Another brief paper of Brioschi's, which I have just seen, is contained in the Atti della R. Acad. dei Lincei, serie terza, Vol. vii, the title is "Le relazioni algebriche fra le funzioni iperellittiche del primo ordine," which, though referring directly only to the double theta-functions, contains some valuable hints as to methods of procedure in the case of functions of more than two arguments. Prof. Cayley gives, in his memoir (page 941), a well known relation between the zero values of the even double theta-functions, viz.

$$\begin{vmatrix} c^2_{12}, & c^2_1, & c^2_6 \\ c^2_9, & -c^2_4, & c^2_3 \\ c^2_2, & -c^2_{15}, & -c^2_8 \end{vmatrix} \div c^2_0 = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

or in my notation

$$\begin{vmatrix} c^2_{13}, & c^2_2, & c^2_7 \\ c^2_{10}, & -c^2_5, & c^2_4 \\ c^2_3, & -c^2_{16}, & -c^2_9 \end{vmatrix} \div c^2_1 = \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

where the $a, b, c, a', \ldots, c''$ constitute a system of coefficients in the transformation between two sets of rectangular coordinates. I have tried to find something similar for the case of the quadruple functions, but so far have not succeeded. It is evident that a relation identical in form with the above cannot exist, inasmuch as there are 9, *i. e.* 3^2 ratios between the even functions in the case of two arguments and 135, which is not a square, ratios between the even functions in the case of four arguments. In volume 94 of Crelle, pp. 74–86, F. Caspary, in a paper entitled *Zur Theorie der Thetafunctionen mit zwei Argumenten,* has obtained the above relation from a more general one; he shows that certain 16 pairs of products of double theta-functions arranged in the form of a determinant of the fourth order satisfy the necessary conditions of an orthogonal substitution. Caspary's fundamental theorem is as follows; using the notation which I have been employing: "If $x_1 x_2$; $y_1 y_2$ denote two pairs of independent arguments, then the sixteen theta products arranged in the following order:

$$\left\{\begin{array}{cccc} \vartheta_{13}(x_1, x_2)\,\vartheta_{13}(y_1, y_2) & \vartheta_3(x_1, x_2)\,\vartheta_3(y_1, y_2) & -\vartheta_{10}(x_1, x_2)\,\vartheta_{10}(y_1, y_2) & -\vartheta_8(x_1, x_2)\,\vartheta_8(y_1, y_2) \\ -\vartheta_7(x_1, x_2)\,\vartheta_7(y_1, y_2) & \vartheta_9(x_1, x_2)\,\vartheta_9(y_1, y_2) & \vartheta_4(x_1, x_2)\,\vartheta_4(y_1, y_2) & -\vartheta_{14}(x_1, x_2)\,\vartheta_{14}(y_1, y_2) \\ \vartheta_2(x_1, x_2)\,\vartheta_2(y_1, y_2) & -\vartheta_{16}(x_1, x_2)\,\vartheta_{16}(y_1, y_2) & \vartheta_5(x_1, x_2)\,\vartheta_5(y_1, y_2) & -\vartheta_{11}(x_1, x_2)\,\vartheta_{11}(y_1, y_2) \\ \vartheta_{12}(x_1, x_2)\,\vartheta_{12}(y_1, y_2) & \vartheta_6(x_1, x_2)\,\vartheta_6(y_1, y_2) & \vartheta_{15}(x_1, x_2)\,\vartheta_{15}(y_1, y_2) & \vartheta_1(x_1, x_2)\,\vartheta_1(y_1, y_2) \end{array}\right\}$$

form the coefficients of a linear substitution in which the sum of the squares of the four new variables is equal to the sum of the squares of the four original variables multiplied by a certain factor." To convert the above arrangement into Prof. Cayley's "current number" notation, it is only necessary to diminish each subscript by unity. The notation adopted by Caspary is that employed by Weierstrass. The arguments x and y are connected with the arguments u and v by the relations

$$x_1 = u_1 + u'_1,\quad y_1 = u_1 - u'_1$$
$$x_1 + y_1 = 2u_1,\quad x_1 - y_1 = 2u'_1, \text{ etc.}$$

It seems to me that one ought to be able by suitably generalizing Caspary's theorem to find the corresponding relation for the quadruple functions. That would seem to be probably the most desirable way of going to work. Caspary passes from the above very general theorem to the particular case which I have cited above, viz. to the relations existing between the ratios of the zero-values of the even double theta-functions. A suitable generalization of his theorem would doubtless lead easily to the corresponding relation between the zero-values of the quadruple functions. This, however, I shall reserve for the second part of this paper.

Sur une formule relative à la théorie des Fonctions d'une variable.

Par M. Hermite.

En restant dans la sphère des questions élémentaires, permettez-moi de vous indiquer une rectification que je viens d'exposer à mes élèves, sur un point traité dans la XII[e] leçon de mon Cour. Il s'agit des applications aux fonctions $\frac{1}{\sin x}$ et $\cot g\, x$ de la formule qui donne l'expression générale des fonctions uniforme $F(x) = \Sigma \left[G_n \left(\frac{1}{x - a_n} \right) + P_\nu(x) \right] + G(x)$. C'est la détermination du terme complémentaire $G(x)$ qui présente une grand difficulté, dont on voit bien la raison, ce terme dépendant essentiellement du polynômes $P_\nu(x)$ qui peuvent être variés d'une infinité de manières. Dans la but d'éviter cette difficulté j'ai fait le remarque que la relation $J = \frac{1}{2i\pi} \int_{(s)} \frac{F(z)\,dz}{z - x} = F(x) - \Sigma\, G_a \left(\frac{1}{x - a} \right)$ donne : $F(x) = \Sigma\, G_a \left(\frac{1}{x - a} \right)$ lorsqu'en agrandissant indéfiniment le contour fermé désigné par S, l'intégrale J relative à ce contour a zéro pour limite. J'ai ensuite pris une circonférence de rayon R ayant son centre à l'origine, ce qui permet d'écrire $J = \frac{\lambda R F(Re^{i\theta})}{Re^{i\theta} - x}$, l'angle θ peuvant avoir une valeur quelconque entre zéro et 2π et λ désignant toujours le facteur de M. Darboux. Il est donc nécessaire pour qu'on puisse conclure de cette expression $J = 0$ que la fonction $F(z)$ tend vers zéro *en tout point* de la circonférence considérée, et c'est ce qui n'arrivera certainement pas dans le cas de $F(z) = \frac{1}{\sin z}$, l'infini étant alors une point singulier essentiel. Et en effet, en posant $z = p + iq$ $\sin z$, croit indéfiniment avec q mais nullement avec le module de $p + iq$, comme il serait nécessaire. L'erreur que

j'ai ainsi commise j'évite aisément si l'on prends pour contour d'intégration au lieu de la circonférence un carré ayant son centre à l'origine des coordonnées. Soit $2a$ le coté de ce carré et posons $F(z) = \frac{F(z)}{z-x}$, on trouve alors:

$$2i\pi J = i\int_{-a}^{+a}[F(a+it) - F(-a+it)]\,dt - \int_{-a}^{+a}[F(ia+t) - F(-ia+t)]\,dt,$$

c'est la formule que je vais employer en supposant en première lieu:

$$F(z) = \frac{1}{z\sin z}.$$

A cet effet je ferai $a = 2m\pi + \alpha$, α étant fixe et m une entier qui croitre indéfiniment, or je remarquerai à l'égard des quantités $\sin(a+it)$, $\sin(ia+t)$, que pour toute valeur de t, le module de la première a pour minimum $\sin\alpha$ le module de la seconde augmentant avec a, au-dela de toute limite. Les expressions suivantes:

$$\int_{-a}^{+a} F(a+it)\,dt = \frac{2a\lambda}{(a+i\tau-x)(a+i\tau)\sin(a+i\tau)}$$

$$\int_{-a}^{+a} F(ia+t)\,dt = \frac{2a\lambda}{(ia+\tau-x)(ia+\tau)\sin(ia+\tau)}$$

où τ est une valeur de t comprise entre $-a$ et $+a$, λ le facteur de M. Darboux, montrent donc que les intégrales ont bien pour limite zéro lorsque a croit indéfiniment. Il en est de même évidemment des deux autres

$$\int_{-a}^{+a} F(-a+it)\,dt \quad \text{et} \int_{-a}^{+a} F(-ia+t)\,dt$$

qui figurent dans l'expression de J et il est ainsi démontré qu'on a $J = 0$ pour a infini. Comme conséquence nous avons la formule

$$\frac{1}{\sin x} = \frac{1}{x^2} + \sum \frac{R_n}{x - n\pi}$$

et le signe Σ s'étendant aux pôles contenues dans la contour d'intégration, on doit attribuer à n toutes les valeurs entières positives et négatives en excluant $n = 0$ qui correspond au pôle double mis à part. Cela étant on trouve $R_n = \frac{(-)^n}{n\pi}$ et nous pouvons écrire $\frac{1}{\sin x} = \frac{1}{x} + \sum \frac{(-1)^n x}{n\pi(x-n\pi)}$

ou encore $$\frac{1}{\sin x} = \frac{1}{x} + \sum (-1)^n \left[\frac{1}{x-n\pi} + \frac{1}{n\pi}\right]$$

puis enfin en recueillant qui correspondent aux valeurs de n égales et de signes contraires:

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_1^\infty {}_n \frac{(-1)^n 2x}{x^2 - n^2\pi^2}.$$

Je passe à la fonction $F(z) = \frac{\operatorname{cotg} z}{z}$, et je considère comme tout-à-l'heure, pour une valeur quelconque de t, les modules de cotg $(a + it)$ et cotg $(ia + t)$.

Or la quantité: $\operatorname{mod}^2 \operatorname{cotg}(a + it) = \frac{\cos 2it + \cos 2a}{\cos 2it - \cos 2a}$ a pour maximum $\frac{1 + \cos 2a}{1 - \cos 2a}$ ou bien l'unité suivant que cos $2a$ est positive ou négative, et quand au module de cotg $(ia + t)$ on sait qu'il est égal à un, pour a infiniment grand. Vous voyez que ces résultats suffisent pour établir que toutes les intégrales composant la valeur de J, ont encore comme précedemment zéro pour limite, on a donc vigoureusement démontré la relation

$$\frac{\operatorname{cotg} x}{x} = \frac{1}{x^2} + \sum \frac{1}{n\pi(x - n\pi)}$$

d'où se tire l'expression ordinaire

$$\operatorname{cotg} x = \frac{1}{x} + \sum_1^\infty {}_n \frac{2x}{x^2 - n^2\pi^2}.$$

Note on a Partition-Series.

By A. Cayley.

Prof. Sylvester, in his paper, A Constructive theory of Partitions, etc., A. M. J. Vol. 5 (1883), has given the following very beautiful formula

$$(1+ax)(1+ax^2)(1+ax^3)\dots$$
$$=1+\frac{1}{1-x}(1+ax^2)xa+\frac{1}{1-x.1-x^2}(1+ax)(1+ax^4)x^5a^2$$
$$+\frac{1}{1-x.1-x^2.1-x^3}(1+ax)(1+ax^2)(1+ax^6)x^{12}a^3+\dots$$

or as this may be written

$$\Omega=1+P+Q(1+ax)+R(1+ax)(1+ax^2)+S(1+ax)(1+ax^2)(1+ax^3)+\dots$$

where

$$P=\frac{(1+ax^2)xa}{\mathbf{1}},\quad Q=\frac{(1+ax^4)x^5a^2}{\mathbf{1.2}},\quad R=\frac{(1+ax^6)x^{12}a^3}{\mathbf{1.2.3}},\quad S=\frac{(1+ax^8)x^{22}a^4}{\mathbf{1.2.3.4}},\text{ etc.,}$$

the heavy figures $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \dots$ of the denominators being, for shortness, written to denote $1-x$, $1-x^2$, $1-x^3$, $1-x^4, \dots$ respectively. The x-exponents 1, 5, 12, 22, ... are the pentagonal numbers $\frac{1}{2}(3n^2-n)$.

To prove this, writing

$$P'=\frac{ax^2}{\mathbf{1}},\quad Q'=\frac{ax^3}{\mathbf{1}}+\frac{a^2x^7}{\mathbf{1.2}},\quad R'=\frac{ax^4}{\mathbf{1}}+\frac{a^2x^9}{\mathbf{1.2}}+\frac{a^3x^{15}}{\mathbf{1.2.3}},$$
$$S'=\frac{ax^5}{\mathbf{1}}+\frac{a^2x^{11}}{\mathbf{1.2}}+\frac{a^3x^{18}}{\mathbf{1.2.3}}+\frac{a^4x^{26}}{\mathbf{1.2.3.4}},\text{ etc.,}$$

where the x-exponents are

$$2;\ 3,\ 3+4;\ 4,\ 4+5,\ 4+5+6;\ 5,\ 5+6,\ 5+6+7,\ 5+6+7+8;\ \text{etc.,}$$

we find without difficulty (see infrà) that

$$1+P=(1+ax)(1+P')$$
$$1+P'+Q=(1+ax^2)(1+Q')$$
$$1+Q'+R=(1+ax^3)(1+R')$$
$$1+R'+S=(1+ax^4)(1+S'),\text{ etc.,}$$

and hence using Ω to denote the sum

$$\Omega=1+P+Q(1+ax)+R(1+ax)(1+ax^2)+S(1+ax)(1+ax^2)(1+ax^3),\text{ etc.,}$$

we obtain successively

$$\Omega\div(1+ax)=1+P'+Q+R(1+ax^2)+S(1+ax^2)(1+ax^3)+\dots$$
$$\Omega\div(1+ax)(1+ax^2)=\pm 1+Q'+R+S(1+ax^3)+T(1+ax^3)(1+ax^4)+\dots$$
$$\Omega\div(1+ax)(1+ax^2)(1+ax^3)=1+R'+S+T(1+ax^4)+\dots$$

and so on. In these equations, on the right-hand sides, the lowest exponent of x is 2, 3, 4, etc., respectively, so that in the limit, the right-hand side becomes

$=1$, or the final equation is $\Omega=(1+ax)(1+ax^2)(1+ax^3)\ldots$, viz., we have the series represented by Ω equal to this infinite product, which is the theorem in question.

One of the foregoing identities is

$$1+R'+S=(1+ax^4)(1+S'),$$

viz. substituting for R', S, S' their values, this is

$$1+\frac{ax^4}{\mathbf{1}}+\frac{a^2x^9}{\mathbf{1.2}}+\frac{a^3x^{15}}{\mathbf{1.2.3}}+\frac{(1+ax^8)a^4x^{22}}{\mathbf{1.2.3.4}}$$
$$=(1+ax^4)\left\{1+\frac{ax^5}{\mathbf{1}}+\frac{a^2x^{11}}{\mathbf{1.2}}+\frac{a^3x^{18}}{\mathbf{1.2.3}}+\frac{a^4x^{26}}{\mathbf{1.2.3.4}}\right\}$$

viz. this equation is

$$-ax^4+\frac{ax^4-ax^5(1+ax^4)}{\mathbf{1}}+\frac{a^2x^9-a^2x^{11}(1+ax^4)}{\mathbf{1.2}}$$
$$+\frac{a^3x^{15}-a^3x^{18}(1+ax^4)}{\mathbf{1.2.3}}+\frac{(1+ax^8)a^4x^{22}-a^4x^{26}(1+ax^4)}{\mathbf{1.2.3.4}},$$

that is

$$0=-ax^4+ax^4-\frac{a^2x^9}{\mathbf{1}}+\frac{a^2x^9}{\mathbf{1}}-\frac{a^3x^{15}}{\mathbf{1.2}}+\frac{a^3x^{15}}{\mathbf{1.2}}-\frac{a^4x^{22}}{\mathbf{1.2.3}}+\frac{a^4x^{22}}{\mathbf{1.2.3}}$$

and in the same way each of the other identities is proved.

Writing $a=-1$ we have $\Omega,=\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\ldots,$

$$=1+P+Q.\mathbf{1}+R.\mathbf{1.2}+S.\mathbf{1.2.3}+\ldots$$

where $\qquad P=-(1+x)x,\ Q=\dfrac{(1+x^2)x^5}{\mathbf{1}},\ R=-\dfrac{(1+x^3)x^{12}}{\mathbf{1.2}},\ldots$

and therefore

$$\mathbf{1.2.3.4}\ldots=1-(1+x)x+(1+x^2)x^5-(1+x^3)x^{12}+\ldots$$

which is Euler's theorem.

It might appear that the identities used in the proof would also for this particular value $a=-1$ lead to interesting theorems, but this is found *not* to be the case: we have

$$P'=\frac{-x^2}{\mathbf{1}},\ Q'=\frac{-x^3}{\mathbf{1}}+\frac{x^7}{\mathbf{1.2}},\ R'=\frac{-x^4}{\mathbf{1}}+\frac{x^9}{\mathbf{1.2}}-\frac{x^5}{\mathbf{1.2.3}},\text{ etc.}$$

but the expressions in terms of these quantities for the products $\mathbf{2.3.4}\ldots$, $\mathbf{3.4}\ldots$, etc., contain denominator factors, and are thus altogether without interest; we have for example

$$\mathbf{2.3.4}\ldots=1+\frac{-x^2+x^5+x^7}{\mathbf{1}}-\frac{(1+x^3)x^{12}}{\mathbf{1}}+\text{etc.}$$

which is with scarcely a change of form the expression obtained from that of the original product $\mathbf{1.2.3.4}\ldots$, by division by $\mathbf{1},=1-x$. And similarly as regards the products $\mathbf{3.4}\ldots$, etc.

CAMBRIDGE, *June*, 1883.

Principles of the Solution of Equations of the Higher Degrees, with Applications.

By George Paxton Young, *Toronto, Canada.*

CONTENTS.

1. Conception of a simple state to which every algebraical expression can be reduced. §6.

2. The particular cognate forms of the generic expression under which a given simplified expression falls are the roots of a rational irreducible equation; and each of the unequal particular cognate forms occurs the same number of times in the series of the cognate forms. §9, 17.

3. Determination of the form which a rational function of the primitive n^{th} root of unity ω_1 and of other primitive roots of unity must have, in order that the substitution of any one of certain primitive n^{th} roots of unity, ω_1, ω_2, ω_3, etc., for ω_1 in the given function may leave the value of the function unaltered. Relation that must subsist among the roots ω_1, ω_2, etc., that satisfy such a condition. §20.

4. If a simplified expression which is the root of a rational irreducible equation of the N^{th} degree involve a surd of the highest rank (§3) not a root of unity, whose index is $\frac{1}{m}$, the denominator of the index being a prime number, N is a multiple of m. But if the simplified root involve no surds that are not roots of unity, and if one of the surds involved in it be the primitive n^{th} root of unity, N is a multiple of a measure of $n-1$. §28.

5. Two classes of solvable equations. §30.

6. The simplified root r_1 of a rational irreducible equation $F(x)=0$ of the m^{th} degree, m prime, which can be solved in algebraical functions, is of the form

$$r_1 = \frac{1}{m}\left(g + \Delta_1^{\frac{1}{m}} + a_1\Delta_1^{\frac{2}{m}} + b_1\Delta_1^{\frac{3}{m}} + \ldots + e_1\Delta_1^{\frac{m-2}{m}} + h_1\Delta_1^{\frac{m-1}{m}}\right);$$

where g is rational, and a_1, b_1, etc., involve only surds subordinate to $\Delta_1^{\frac{1}{m}}$. §38, 47.

7. The equation $F(x)=0$ has an auxiliary equation of the $(m-1)^{\text{th}}$ degree. §35, 52.

8. If the roots of the auxiliary be $\Delta_1, \delta_2, \delta_3, \ldots, \delta_{m-1}$, the $m-1$ expressions in each of the groups

$$\begin{array}{lll} \Delta_1^{\frac{1}{m}}\delta_{m-1}^{\frac{1}{m}}, & \delta_2^{\frac{1}{m}}\delta_{m-2}^{\frac{1}{m}}, \ldots, & \delta_{m-1}^{\frac{1}{m}}\Delta_1^{\frac{1}{m}}, \\ \Delta_1^{\frac{2}{m}}\delta_{m-2}^{\frac{1}{m}}, & \delta_2^{\frac{2}{m}}\delta_{m-4}^{\frac{1}{m}}, \ldots, & \delta_{m-1}^{\frac{2}{m}}\delta_2^{\frac{1}{m}}, \\ \Delta_1^{\frac{3}{m}}\delta_{m-3}^{\frac{1}{m}}, & \delta_2^{\frac{3}{m}}\delta_{m-6}^{\frac{1}{m}}, \ldots, & \delta_{m-1}^{\frac{3}{m}}\delta_3^{\frac{1}{m}}, \end{array}$$

and so on, are the roots of a rational equation of the $(m-1)^{\text{th}}$ degree. The $\frac{m-1}{2}$ terms

$$\Delta_1^{\frac{1}{m}}\delta_{m-1}^{\frac{1}{m}}, \quad \delta_2^{\frac{1}{m}}\delta_{m-2}^{\frac{1}{m}}, \ldots, \quad \delta_{\frac{m-1}{2}}^{\frac{1}{m}}\delta_{\frac{m+1}{2}}^{\frac{1}{m}},$$

are the roots of a rational equation of the $\left(\frac{m-1}{2}\right)^{\text{th}}$ degree. §39, 44, 55.

9. Wider generalization. §45, 57.

10. When the equation $F(x)=0$ is of the first class, the auxiliary equation of the $(m-1)^{\text{th}}$ degree is irreducible. §35. Also the roots of the auxiliary are rational functions of the primitive m^{th} root of unity. §36. And, in the particular case when the equation $F(x)=0$ is the reducing Gaussian equation of the m^{th} degree to the equation $x^n-1=0$, each of the $\frac{m-1}{2}$ expressions, $\Delta_1^{\frac{1}{m}}\delta_{m-1}^{\frac{1}{m}}$, $\delta_2^{\frac{1}{m}}\delta_{m-2}^{\frac{1}{m}}$, &c., has the rational value n. §41. Numerical verification, §42.

11. Solution of the Gaussian. §43.

12. Analysis of solvable irreducible equations of the fifth degree. The auxiliary biquadratic either is irreducible, or has an irreducible sub-auxiliary of the second degree, or has all its roots rational. The three cases considered separately. Deduction of Abel's expression for the roots of a solvable quintic. §58–74.

PRINCIPLES.

§1. It will be understood that the surds appearing in the present paper have *prime numbers* for the denominators of their indices, unless where the contrary is expressly stated. Thus, $2^{\frac{1}{15}}$ may be regarded as $h^{\frac{1}{5}}$, a surd with the index $\frac{1}{5}$, h being $2^{\frac{1}{3}}$. It will be understood also that no surd appears in the denominator of a fraction. For instance, instead of $\frac{2}{1+\sqrt{-3}}$ we should write $\frac{1-\sqrt{-3}}{2}$. When a surd is spoken of as occurring in an algebraical expression,

it may be present in more than one of its powers, and need not be present in the first.

§2. In such an expression as $\sqrt{2}+(1+\sqrt{2})^{\frac{1}{3}}$, $\sqrt{2}$ is *subordinate* to the *principal* surd $(1+\sqrt{2})^{\frac{1}{3}}$, the latter being the only principal surd in the expression.

§3. A surd that has no other surd subordinate to it may be said to be *of the first rank;* and the surd $h^{\frac{1}{c}}$, where h involves a surd of the $(a-1)^{\text{th}}$ rank, but none of a higher rank, may be said to be *of the a^{th} rank.* In estimating the rank of a surd, the denominators of the indices of the surds concerned are always supposed to be prime numbers. Thus, $3^{\frac{1}{6}}$ is a surd of the second rank.

§4. An algebraical expression in which $\Delta_1^{\frac{1}{m}}$ is a principal (see §2) surd may be arranged according to the powers of $\Delta_1^{\frac{1}{m}}$ lower than the m^{th}, thus,

$$\frac{1}{m}\left(g_1+k_1\Delta_1^{\frac{1}{m}}+a_1\Delta_1^{\frac{2}{m}}+b_1\Delta_1^{\frac{3}{m}}+\ldots+e_1\Delta_1^{\frac{m-2}{m}}+h_1\Delta_1^{\frac{m-1}{m}}\right); \tag{1}$$

where g_1, k_1, a_1, etc., are clear of $\Delta_1^{\frac{1}{m}}$.

§5. If an algebraical expression r_1, arranged as in (1), be zero, while the coefficients g_1, k_1, etc., are not all zero, an equation

$$\omega\Delta_1^{\frac{1}{m}}=l_1 \tag{2}$$

must subsist; where ω is an m^{th} root of unity; and l_1 is an expression involving only such surds exclusive of $\Delta_1^{\frac{1}{m}}$ as occur in r_1. For, let the first of the coefficients h_1, e_1, etc., proceeding in the order of the descending powers of $\Delta_1^{\frac{1}{m}}$, that is not zero, be n_1, the coefficient of $\Delta_1^{\frac{s}{m}}$. Then we may put

$$mr_1=n_1\{f(\Delta_1^{\frac{1}{m}})\}=n_1\Delta_1^{\frac{s}{m}}+\text{etc.}=0\,.$$

Because $\Delta_1^{\frac{1}{m}}$ is a root of each of the equations $f(x)=0$ and $x^m-\Delta_1=0$, $f(x)$ and $x^m-\Delta_1$ have a common measure. Let their H. C. M., involving only such surds as occur in $f(x)$ and $x^m-\Delta_1$, be $\phi(x)$. Then, because $\phi(x)$ is a measure of $x^m-\Delta_1$, the roots of the equation

$$\phi(x)=x^c+p_1x^{c-1}+p_2x^{c-2}+\text{etc.}=0$$

are $\Delta_1^{\frac{1}{m}}$, $\omega_1\Delta_1^{\frac{1}{m}}$, $\omega_2\Delta_1^{\frac{1}{m}}$, $\ldots$, $\omega_{c-1}\Delta_1^{\frac{1}{m}}$; where ω_1, ω_2, etc., are distinct primitive m^{th} roots of unity. Therefore $\Delta_1^{\frac{c}{m}}(\omega_1\omega_2\ldots)(-1)^c=p_c$. Now c is a whole number less than m but not zero; and, by §1, m is prime. Therefore there are whole numbers n and h such that

$$\Delta_1^{\frac{cn}{m}}(\omega_1\omega_2\ldots)^n(-1)^{cn} = \Delta_1^{\frac{1}{m}}\Delta_1^h(\omega_1\omega_2\ldots)^n(-1)^{cn} = p_c^n.$$

Therefore, if $(\omega_1\omega_2\ldots)^n = \omega$, and $l_1\Delta_1^h(-1)^{cn} = p_c^n$, $\omega\Delta_1^{\frac{1}{m}} = l_1$.

§6. Let r_1 be an algebraical expression in which no root of unity having a rational value occurs in the surd form $1^{\frac{1}{m}}$. Also let there be in r_1 no surd $\Delta_1^{\frac{1}{m}}$ not a root of unity, such that

$$\Delta_1^{\frac{1}{m}} = e_1, \tag{3}$$

where e_1 is an expression involving no surds of so high a rank as $\Delta_1^{\frac{1}{m}}$, except such as either are roots of unity, or occur in r_1 being at the same time distinct from $\Delta_1^{\frac{1}{m}}$. The expression r_1 may then be said to have been *simplified* or to be *in a simple state.*

§7. Some illustrations of the definition in §6 may be given. The root $8^{\frac{1}{3}}$ cannot occur in a simplified expression r_1; for its value is 2ω, ω being a third root of unity; but the equation $8^{\frac{1}{3}} = 2\omega$ is of the inadmissible type (3). Again, the root $\sqrt{5}$ cannot occur in a simplified expression; for, ω_1 being a primitive fifth root of unity, $\sqrt{5} = 2(\omega_1 + \omega_1^4) + 1$; an equation of the type (3). Once more, a root of the cubic equation $x^3 - 3x - 4 = 0$, in the form $(2 + \sqrt{3})^{\frac{1}{3}} + (2 - \sqrt{3})^{\frac{1}{3}}$, is not in a simple state, because $(2 - \sqrt{3})^{\frac{1}{3}} = (2 - \sqrt{3})(2 + \sqrt{3})^{\frac{2}{3}}$.

§8. Let

$$p_1\Delta_1^{\frac{m-1}{m}} + p_2\Delta_1^{\frac{m-2}{m}} + \ldots + p_m = 0; \tag{4}$$

where $\Delta_1^{\frac{1}{m}}$ is a surd occurring in the simplified expression r_1; and p_1, p_2, etc., involve no surds of so high a rank as $\Delta_1^{\frac{1}{m}}$, except such as either are roots of unity, or occur in r_1 being at the same time distinct from $\Delta_1^{\frac{1}{m}}$. The coefficients p_1, p_2, etc., must be zero separately. For, by §5, if they were not, we should have $\omega\Delta_1^{\frac{1}{m}} = l_1$, ω being an m^{th} root of unity, and l_1 involving only surds in (4) distinct from $\Delta_1^{\frac{1}{m}}$; an equation of the inadmissible type (3).

§9. The expression r_1 being in a simple state, we may use R as a generic symbol to include the various particular expressions, say r_1, r_2, r_3, etc., obtained by assigning all their possible values to the surds involved in r_1, with the restriction that, where the base of a surd is unity, the rational value of the surd is not to be taken into account. These particular expressions, not necessarily all unequal, may be called *the particular cognate forms of R*. For instance, if $r_1 = 1^{\frac{1}{3}}$, R has two particular cognate forms, the rational value of the third root of unity not being counted. If $r_1 = (1 + \sqrt{2})^{\frac{1}{3}}$, R has six particular cognate

forms all unequal. Should $r_1 = (2+\sqrt{3})^{\frac{1}{3}} + (2-\sqrt{3})(2+\sqrt{3})^{\frac{2}{3}}$, R has six particular cognate forms, but only three unequal, each of the unequal forms occurring twice.

§10. Proposition I. An algebraical expression r_1 can always be brought to a simple state.

For r_1 may be cleared of all surds such as $1^{\frac{1}{m}}$ having a rational value. Suppose that r_1 then involves a surd $\Delta_1^{\frac{1}{m}}$, not a root of unity, by means of which an equation such as (3) can be formed. Substitute for $\Delta_1^{\frac{1}{m}}$ in r_1 its value e_1 as thus given. The result will be to eliminate $\Delta_1^{\frac{1}{m}}$ from r_1 without introducing into the expression any *new* surd as high in rank as $\Delta_1^{\frac{1}{m}}$, and at the same time not a root of unity. By continuing to make all the eliminations of this kind that are possible, we at last reach a point where no equation of the type (3) can any longer be formed. Then because, by the course that has been pursued, no roots of the form $1^{\frac{1}{m}}$ having a rational value have been left in r_1, r_1 is in a simple state.

§11. It is known that, if N be any whole number, the equation whose roots are the primitive N^{th} roots of unity is rational and irreducible.

§12. Let N be the continued product of the distinct prime numbers n, a, b, etc. Let ω_1 be a primitive n^{th} root of unity, θ_1 a primitive a^{th} root of unity, and so on. Let ω represent any one indifferently of the primitive n^{th} roots of unity, θ any one indifferently of the primitive a^{th} roots of unity, and so on. Let $f(\omega_1, \theta_1, \text{etc.})$ be a rational function of ω_1, θ_1, etc. Then a corollary from §11 is, that if $f(\omega_1, \theta_1, \text{etc.}) = 0$, $f(\omega, \theta, \text{etc.}) = 0$. For t_1 being a primitive N^{th} root of unity, and t representing any one indifferently of the primitive N^{th} roots of unity, we may put

$$f(\omega_1, \theta_1, \text{etc.}) = a_1 t_1^{N-1} + a_2 t_1^{N-2} + \text{etc.} = 0,$$
$$\text{and } f(\omega, \theta, \text{etc.}) = a_1 t^{N-1} + a_2 t^{N-2} + \text{etc.};$$

where the coefficients a_1, a_2, etc., are rational. Should these coefficients be all zero, $f(\omega, \theta, \text{etc.}) = 0$. Should they not be all zero, let a_r be the first that is not zero. Then we may put

$$f(\omega_1, \theta_1, \text{etc.}) = a_r\{\phi(t_1)\} = a_r t_1^{N-r} + \text{etc.} = 0.$$

Therefore, t_1 is a root of the rational equation $\phi(x) = 0$, being at the same time a root of the rational (see §11) equation $\psi(x) = 0$, whose roots are the primitive N^{th} roots of unity. Hence $\psi(x)$ and $\phi(x)$ have a common measure. But by §11, $\psi(x)$ is irreducible. Therefore it is a measure of $\phi(x)$; and the roots of

the equation $\psi(x)=0$ are roots of the equation $\phi(x)=0$. Therefore, $f(\omega, \theta, \text{etc.}) = a_r\{\phi(t)\}=0$.

§13. Another corollary is, that if

$$f(\omega_1, \theta_1, \text{etc.}) = h_1\omega_1^{N-1} + h_2\omega_1^{N-2} + \ldots + h_n = 0,$$

where h_1, h_2, etc., are clear of ω_1, the coefficients h_1, h_2, etc., are all equal to one another. For, by §12, because $f(\omega_1, \theta_1, \text{etc.})=0$, $f(\omega, \theta_1, \text{etc.})=0$. Therefore $\omega\{f(\omega, \theta_1, \text{etc.})\}=0$. In $\omega\{f(\omega, \theta_1, \text{etc.})\}$ give ω successively its $n-1$ different values. Then, by addition,

$nh_1 = h_1 + h_2 + \ldots + h_n$. Similarly, $nh_2 = h_1 + h_2 + \ldots + h_n \therefore h_1 = h_2$.
In like manner all the terms h_1, h_2, etc., are equal to one another.

§14. PROPOSITION II. If the simplified expression r_1, one of the particular cognate forms of R, be a root of the rational equation $F(x)=0$, all the particular cognate forms of R are roots of that equation.

For, let r_2 be a particular cognate form of R. By §12, the law to be established holds when there are no surds in r_1 that are not roots of unity. It will be kept in view that, according to §1, when roots of unity are spoken of, such roots are meant as $1^{\frac{1}{m}}$, m being a prime number. Assume the law to have been found good for all expressions that do not involve more than $n-1$ distinct surds that are not roots of unity; then, making the hypothesis that r_1 involves not more than n distinct surds that are not roots of unity, the law can be shown still to hold; in which case it must hold universally. For, let $\Delta_1^{\frac{1}{m}}$, not a root of unity, be a surd of the highest rank (see §3) in r_1. Then $F(r_1)$ may be taken to be the expression (1), and $F(r_2)$ to be the expression formed from (1) by selecting particular values of the surds involved under the restriction specified in §9. In passing from r_1 to r_2, let $\Delta_1^{\frac{1}{m}}$, a_1, etc., become respectively $\Delta_2^{\frac{1}{m}}$, a_2, etc.

Then $$m\{F(r_1)\} = h_1\Delta_1^{\frac{m-1}{m}} + e_1\Delta_1^{\frac{m-2}{m}} + \text{etc.} = 0,$$

$$\text{and } m\{F(r_2)\} = h_2\Delta_2^{\frac{m-1}{m}} + e_2\Delta_2^{\frac{m-2}{m}} + \text{etc.}$$

By §8, because r_1 is in a simple state, and $F(r_1)=0$, the coefficients h_1, e_1, etc., are zero separately. But h_1 is clear of the surd $\Delta_1^{\frac{1}{m}}$. It therefore does not involve more than $n-1$ distinct surds that are not roots of unity. Therefore, on the assumption on which we are proceeding, because $h_1=0$, $h_2=0$. In like manner, $e_2=0$, and so on. Therefore $F(r_2)=0$.

§15. *Cor.* Let the simplified expression r_1 be the root of an equation $F(x)=0$ whose coefficients involve certain surds $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc., that have the

same determinate values in r_1 as in $F(x)$. Then, if r_2 be a particular cognate form of R in which the surds $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, r_2 is a root of the equation $F(x)=0$. For, $F(r_1)=0$. Therefore, by the Proposition, $F(R)=0$. Let R, restricted by the condition that the surds $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, be R'. Then $F(R')=0$. A particular case of this is $F(r_2)=0$. The corollary established simply means that the surds $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc., may be taken to be rational for the purpose in hand.

§16. The simplified expression r_1 being one of the particular cognate forms of R, let

$$r_1,\ r_a,\ \text{etc.} \tag{5}$$

be the entire series of the particular cognate forms of R, not necessarily unequal to one another. Then, if the equation whose roots are the terms in (5) be $X=0$, X is rational. In like manner, if those particular cognate forms of R, not necessarily unequal, that are obtained when certain surds $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, be

$$r_1,\ r_e,\ \text{etc.} \tag{6}$$

and if the equation whose roots are the terms in (6) be $X'=0$, X' involves only surds found in the series $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc. This is substantially proved by Legendre in his Théorie des Nombres, §487, third edition.

§17. Proposition III. The unequal particular cognate forms of R, the generic expression under which the simplified expression r_1 falls, are the roots of a rational irreducible equation; and each of the unequal particular cognate forms occurs the same number of times in the series of the cognate forms.

As in §16, let the entire series of the particular cognate forms of R be the terms in (5), the equation that has these terms for its roots being $X=0$. By §16, X is rational. Should X not be irreducible, it has a rational irreducible factor, say $F(x)$, such that r_1 is a root of the equation $F(x)=0$. By Prop. II, because r_1 is in a simple state, all the terms in (5) are roots of the equation $F(x)=0$, while at the same time, because $F(x)$ is a factor of X, all the roots of the equation are terms in (5). And the equation $F(x)=0$, being irreducible, has no equal roots. Therefore its roots are the unequal terms in (5). Should $F(x)$ not be identical with X, put $X=\{F(x)\}\{\phi(x)\}$. Because X and $F(x)$ are rational, $\phi(x)$ is rational. Then, since $\phi(x)$ is a measure of X, and the equation $F(x)=0$ has for its roots the unequal roots of the equation $X=0$, the equations

$F(x)=0$ and $\phi(x)=0$ have a root in common. Consequently, since $F(x)$ is irreducible, it is a measure of $\phi(x)$. Therefore $\{F(x)\}^2$ is a measure of X. Going on in this way we ultimately get $X=\{F(x)\}^N$; which means that each of the particular cognate forms of R has its value repeated N times in the series of the particular cognate forms.

§18. *Cor.* 1. The series (6) consisting of those particular cognate forms of R in which certain surds $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc., retain the determinate values belonging to them in r_1, each of the unequal terms in (6) occurs the same number of times in (6); and the unequal terms in (6) are the roots of an irreducible equation whose coefficients involve only surds found in the series $z_1^{\frac{1}{n}}$, $u_1^{\frac{1}{s}}$, etc. Should X' not be irreducible, by which in such a case is meant incapable of being broken into lower factors involving only surds occurring in X', let it have the irreducible factor X''. That is to say, X'' involves only surds occurring in X', and has itself no lower factor involving only surds that occur in X''. We may take r_1 to be a root of the equation $X''=0$. Then, by Cor. Prop. II, all the terms in (6) are roots of that equation, all the roots of the equation being at the same time terms in (6). And the equation $X''=0$ being irreducible, has no equal roots. Therefore its roots are the unequal terms in (6). Put $X'=(X'')(X''')$. Then, by the line of reasoning followed in the Proposition, X''' has a measure identical with X''. And so on. Ultimately $X'=(X'')^N$.

§19. *Cor.* 2. If r_2, one of the particular cognate forms of R, be zero, all the particular cognate forms of R are zero. For, by the proposition, the particular cognate forms of R are the roots of a rational irreducible equation $F(x)=0$. And r_2, one of the roots of that equation, is zero, but the only rational irreducible equation that has zero for a root is $x=0$. Therefore $F(x)=x=0$. In fact, in the case supposed, the simplified expression r_1 is zero, and R has no particular cognate forms distinct from r_1.

§20. PROPOSITION IV. Let N be the continued product of the distinct prime numbers n, a, etc. Let ω_1 be a primitive n^{th} root of unity, θ_1 a primitive a^{th} root of unity, and so on. Then if the equation

$$F(x)=x^d+b_1x^{d-1}+b_2x^{d-2}+\text{etc.}=0$$

be one in which the coefficients b_1, b_2, etc., are rational functions of ω_1, θ_1, etc., and if all the primitive n^{th} roots of unity, which, when substituted for ω_1 in $F(x)$, leave $F(x)$ unaltered, be

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_s, \tag{7}$$

the series (7) either consists of a single term or it is made up of a cycle of primitive n^{th} roots of unity,

$$\omega_1,\ \omega_1^{\lambda},\ \omega_1^{\lambda^2},\ \ldots,\ \omega_1^{\lambda^{s-1}}; \tag{8}$$

that is to say, no term in (8) after the first is equal to the first, but $\omega_1^{\lambda^s} = \omega_1$. Also, if (let it be kept in view that n is prime) the cycle that contains all the primitive n^{th} roots of unity be

$$\omega_1,\ \omega_1^{\beta},\ \omega_1^{\beta^2},\ \ldots,\ \omega_1^{\beta^{n-2}}, \tag{9}$$

and if C_1 be the sum of the terms in the cycle (8), the form of $F(x)$ is

$$F(x) = x^d - (p_1 C_1 + p_2 C_2 + \ldots + p_m C_m)x^{d-1} + (q_1 C_1 + q_2 C_2 + \text{etc.})x^{d-2} + \text{etc.} \tag{10}$$

where each of the expressions in the series C_1, C_2, C_3, etc., is what the immediately preceding term becomes by changing ω_1 into ω_1^{β}, C_m through this change becoming C_1; and p_1, p_2, q_1, etc., are clear of ω_1.

For, assuming that there is a term ω_2 in (7) additional to ω_1, we may take ω_2 to be the first term in (9) after ω_1 that occurs in (7); and it may be considered to be $\omega_1^{\beta^m}$, which may be otherwise written ω_1^{λ}. Then, if $F(x)$ be written $\phi(\omega_1)$, we have by hypothesis $\phi(\omega_1) = \phi(\omega_1^{\lambda})$. Therefore, by §12, changing ω_1 into ω_1^{λ}, $\phi(\omega_1^{\lambda}) = \phi(\omega_1^{\lambda^2})$. Therefore $\phi(\omega_1) = \phi(\omega_1^{\lambda^2})$. And thus ultimately $\phi(\omega_1) = \phi(\omega_1^{\lambda^z})$, or $\phi(\omega_1) = \phi(\omega_1^{\beta^{mz}})$, z being any whole number positive or negative. But $\omega_1^{\lambda^z}$ includes all the terms in (8). Therefore each of these terms is a term in (7). Suppose if possible that there is a term in (7), say $\omega_1^{\beta^h}$, which does not occur in (8). Then, just as we deduced $\phi(\omega_1) = \phi(\omega_1^{\beta^{mz}})$ from the equation $\phi(\omega_1) = \phi(\omega_1^{\beta^m})$, we can, because still farther $\phi(\omega_1) = \phi(\omega_1^{\beta^h})$, deduce $\phi(\omega_1) = \phi(\omega_1^{\beta^{mz+hu}})$. Because $\omega_1^{\beta^h}$ lies outside the cycle (8), h is not a multiple of m. And it is not less than m, because $\omega_1^{\beta^m}$ is the first term in (9) after ω_1, which, when substituted for ω_1 in $\phi(\omega_1)$, leaves $\phi(\omega_1)$ unaltered. Therefore $h = qm + v$, where q and v are whole numbers, and v is less than m but not zero. Put

$$z = -(h+q),\ \text{and}\ u = m + 1 \quad \therefore mz + hu = v \quad \therefore \phi(\omega_1) = \phi(\omega_1^{\beta^v});$$

which, because v is less than m but not zero, and $\omega_1^{\beta^m}$ is the first term in (9) after ω_1 which, when substituted for ω_1 in $\phi(\omega_1)$, leaves $\phi(\omega_1)$ unaltered, is impossible. Hence no term in (7) lies outside the cycle (8), while it has also been shown that all the terms in (8) are terms in (7). Therefore the terms in (7) are identical with those constituting the cycle (8). We have now to determine the form of $F(x)$. The expressions, C_1, C_2, etc., taken together are the sum of the terms in (9). Therefore

$$C_1 + C_2 + \ldots + C_m = -1. \tag{11}$$

Because (9) contains all the primitive n^{th} roots of unity, we may put

$$F(x) = x^d - \{p + (p+p_1)\omega_1 + (p+p_2)\omega_1^{\beta} + \text{etc.}\}x^{d-1} + \text{etc.}; \tag{12}$$

where p, p_1, etc., are clear of ω_1. But $F(x)$ remains unaltered when ω_1 is changed into $\omega_1^{\beta m}$. Therefore

$$F(x) = x^d - \{ p + (p + p_1)\omega_1^{\beta m} + \text{etc.} \} x^{d-1} + \text{etc.} \tag{13}$$

Therefore, equating the coefficients of x^{d-1} in (12) and (13),

$$(p - p_1) + \ldots + (p_{m+1} - p_1)\omega_1^{\beta m} + \text{etc.} = 0.$$

Here, by §13, the coefficients of the different powers of ω_1 have all the same value. And one of them, $p - p_1$, is zero. Therefore $p_{m+1} = p_1$. That is to say, the coefficient of $\omega_1^{\beta m}$ or ω_1^{λ} is the same as that of ω_1. In like manner the coefficients of all the terms in (8) are the same. Therefore one group of the terms that together make up the coefficient of x^{d-1} in (12) is properly represented by $-(p + p_1)C_1$. In the same way another group is properly represented by $-(p + p_2)C_2$, and so on. Hence

$$F(x) = x^d - \{ p + (p + p_1)C_1 + (p + p_2)C_2 + \text{etc.} \} x^{d-1} + \text{etc.}$$

And by (11) this is equivalent to (10). The form of $F(x)$ has been deduced on the assumption that the series (7) contains more than one term; but, should the series (7) consist of a single term, the result obtained would still hold good, only in that case each of the expressions C_1, C_2, etc., would be a primitive n^{th} root of unity.

§21. A simplified expression will not cease to be in a simple state, if we suppose that any surd that can be eliminated from it, without the introduction of any new surd, has been eliminated.

§22. PROPOSITION V. In the simplified expression r_1, one of the particular cognate forms of R, modified according to §21, let the surd $\Delta_1^{\frac{1}{m}}$ of the highest rank be not a root (see §1) of unity. Then, if the particular cognate forms of R obtained by changing $\Delta_1^{\frac{1}{m}}$ in r_1 successively into the different m^{th} roots of the determinate base Δ_1, be

$$r_1, r_2, \ldots, r_m, \tag{14}$$

these terms are all unequal.

For the terms in (14) are all the particular cognate forms of R obtained when we allow all the surds in r_1 except $\Delta_1^{\frac{1}{m}}$ to retain the determinate values belonging to them in r_1. Therefore, by Cor. 1, Prop. III, each of the unequal terms in (14) has its value repeated the same number of times in that series. Let u be the number of the unequal terms in (14), and let each occur c times. Then $uc = m$. Suppose if possible that $u = 1$. This means that all the terms in (14) are equal. Therefore, r_1 being the expression (1),

$$mr_1 = r_1 + r_2 + \ldots + \text{etc.} = g_1.$$

Therefore the surd $\Delta_1^{\frac{1}{m}}$ can be eliminated from r_1 without the introduction of any new surd; which, by §21, is impossible. Therefore u is not unity. But, by §1, m is a prime number. And $m = uc$. Therefore $c = 1$ and $u = m$. This means that all the terms in (14) are unequal.

§23. *Cor.* 1. Let r_{a+1} be any one of the particular cognate forms of R; and let $\Delta_{a+1}^{\frac{1}{m}}$, h_{a+1}, etc., be respectively what $\Delta_1^{\frac{1}{m}}$, h_1, etc., become in passing from r_1 to r_{a+1}. Also let the m particular cognate forms of R, obtained by changing $\Delta_{a+1}^{\frac{1}{m}}$ in r_{a+1} successively into the different m^{th} roots of Δ_{a+1}, be

$$r_{a+1},\ r_{a+2},\ \ldots,\ r_{a+m}. \tag{15}$$

These terms are all unequal. For, because $\Delta_1^{\frac{1}{m}}$ is a principal surd in r_1, and r_2 is what r_1 becomes when $\Delta_1^{\frac{1}{m}}$ is changed into a surd whose value is $\omega_1 \Delta_1^{\frac{1}{m}}$, ω_1 being a primitive m^{th} root of unity, the view may be taken that r_2 involves no surds additional to those found in r_1, except the primitive m^{th} root of unity ω_1. Therefore $r_1 - r_2$ involves no surds distinct from primitive m^{th} roots of unity that are not found in the simplified expression r_1. Therefore $r_1 - r_2$ is in a simple state. Let r_{a+2} be what r_{a+1} becomes by changing $\Delta_{a+1}^{\frac{1}{m}}$ into $\omega_1 \Delta_{a+1}^{\frac{1}{m}}$. Then $r_{a+1} - r_{a+2}$ is a particular cognate form of the generic expression under which the simplified expression $r_1 - r_2$ falls. Therefore $r_{a+1} - r_{a+2}$ cannot be zero; for, if it were, $r_1 - r_2$ would, by Cor. 2, Prop. III, be zero; which, by the proposition, is impossible. Hence the first two terms in (15) are unequal. In like manner all the terms in (15) are unequal.

§24. *Cor.* 2. Let $X_1 = 0$ be the equation whose roots are the terms in (14). When X_1 is modified according to §21, it is, by §16, clear of the surd $\Delta_1^{\frac{1}{m}}$. Should it involve any surds that are not roots of unity, take $z_1^{\frac{1}{c}}$ a surd of the highest rank not a root of unity in X_1; and, when $z_1^{\frac{1}{c}}$ is changed successively into the different c^{th} roots of the determinate base z_1, let

$$X_1,\ X_1',\ X_1'',\ \ldots,\ X_1^{(c-1)}, \tag{16}$$

be respectively what X_1 becomes. Any term in (16), as X_1', being selected, the m roots of the equation $X_1' = 0$ are unequal particular cognate forms of R. For, $z_2^{\frac{1}{c}}$ being a c^{th} root of z_1 distinct from $z_1^{\frac{1}{c}}$, let r_{a+1} be what r_1 becomes when $z_1^{\frac{1}{c}}$ becomes $z_2^{\frac{1}{c}}$; the expressions $\Delta_1^{\frac{1}{m}}$, h_1, etc., at the same time becoming $\Delta_{a+1}^{\frac{1}{m}}$, h_{a+1}, etc. Then we may put

$$X_1 = x^m + (bz_1^{\frac{c-1}{c}} + dz_1^{\frac{c-2}{c}} + \text{etc.})x^{m-1} + \text{etc}; \tag{17}$$

where b, d, etc., are clear of $z_1^{\frac{1}{c}}$. Therefore, because r_1 is a root of the equation $X_1 = 0$,

$$\{\tfrac{1}{m}(h_1\Delta_1^{\frac{m-1}{m}} + \text{etc.})\}^m + (bz_1^{\frac{c-1}{c}} + dz_1^{\frac{c-2}{c}} + \text{etc.})\{\tfrac{1}{m}(h_1\Delta_1^{\frac{m-1}{m}} + \text{etc.})\}^{m-1} + \text{etc.} = 0.$$

All the surds in this equation occur in the simplified expression r_1. Therefore, by Prop. II,

$$\{\tfrac{1}{m}(h_{a+1}\Delta_{a+1}^{\frac{m-1}{m}} + \text{etc.})\}^m + (bz_2^{\frac{c-1}{c}} + dz_2^{\frac{c-2}{c}} + \text{etc.})\{\tfrac{1}{m}(h_{a+1}\Delta_{a+1}^{\frac{m-1}{m}} + \text{etc.})\}^{m-1} + \text{etc.} = 0.$$

Therefore $\frac{1}{m}(h_{a+1}\Delta_{a+1}^{\frac{m-1}{m}} + \text{etc.})$ or r_{a+1} is a root of the equation

$$X_1' = x^m + (bz_2^{\frac{c-1}{c}} + \text{etc.})x^{m-1} + \text{etc.} = 0. \tag{18}$$

Therefore also, by Cor. Prop. II, all the terms in (15) are roots of that equation. And, by Cor. 1, the terms in (15) are all unequal. Therefore the equation $X_1' = 0$ has m unequal particular cognate forms of R for its roots.

§25. *Cor.* 3. No two of the expressions in (16), as X_1 and X_1', are identical with one another. For, in order that X_1 and X_1' might be identical, the coefficients of the several powers of x in X_1 would need to be equal to those of the corresponding powers of x in X_1'; but, if one of the coefficients of X_1 be selected in which $z_1^{\frac{1}{c}}$ is present, this coefficient can be shown to be unequal to the corresponding coefficient in X_1' in the same way in which the terms in (15) were proved to be all unequal.

§26. *Cor.* 4. Any two of the terms in (16), as X_1 and X_1', being selected, the equations $X_1 = 0$ and $X_1' = 0$ have no root in common. For, suppose, if possible, that these equations have a root in common. Taking the forms of X_1 and X_1' in (17) and (18), since r_1 is a root of the equation $X_1' = 0$,

$$r_1^m + (bz_2^{\frac{c-1}{c}} + \text{etc.})r_1^{m-1} + \text{etc.} = 0. \tag{19}$$

All the surds in this equation except $z_2^{\frac{1}{c}}$ occur in r_1. It is impossible that $z_2^{\frac{1}{c}}$ can occur in r_1; for, $z_1^{\frac{1}{c}}$ occurs in r_1; and $z_2^{\frac{1}{c}} = \theta_1 z_1^{\frac{1}{c}}$, θ_1 being a primitive c^{th} root of unity; but this equation, if both $z_1^{\frac{1}{c}}$ and $z_2^{\frac{1}{c}}$ occurred in r_1, would be of the inadmissible type (3). Since $z_2^{\frac{1}{c}}$ does not occur in r_1, it is a principal (see §2) surd in (19). We may, therefore, keeping in view that r_1 is the expression (1) in which $\Delta_1^{\frac{1}{m}}$ is a principal surd, arrange (19) thus,

$$\phi(\Delta_1^{\frac{1}{m}}) = \Delta_1^{\frac{m-1}{m}}(p_1 z_2^{\frac{c-1}{c}} + p_2 z_2^{\frac{c-2}{c}} + \text{etc.}) + \Delta_1^{\frac{m-2}{m}}(q_1 z_2^{\frac{c-1}{c}} + q_2 z_2^{\frac{c-2}{c}} + \text{etc.}) + \text{etc.} = 0; \tag{20}$$

where p_1, q_1, etc., are clear of $z_2^{\frac{1}{c}}$. Then, ω_1 being a primitive m^{th} root of unity

such that, by changing $\Delta_1^{\frac{1}{m}}$ into the m^{th} root of Δ_1 whose value is $\omega_1\Delta_1^{\frac{1}{m}}$, r_1 becomes r_2,

$$\phi(\omega_1\Delta_1^{\frac{1}{m}}) = \omega_1^{m-1}\Delta_1^{\frac{m-1}{m}}(p_1 z_2^{\frac{c-1}{c}} + \text{etc.}) + \omega_1^{m-2}\Delta_1^{\frac{m-1}{m}}(q_1 z_2^{\frac{c-1}{c}} + \text{etc.}) + \text{etc.} \quad (21)$$

The coefficients of the several powers of $\Delta_1^{\frac{1}{m}}$ in $\phi(\Delta_1^{\frac{1}{m}})$ cannot be all zero; for, if they were, we should have, from (21), $\phi(\omega_1\Delta_1^{\frac{1}{m}}) = 0$. This means that r_2 is a root of the equation $X_1' = 0$. But in like manner all the terms in (14) would be roots of that equation, and X_1' would be identical with X; which, by Cor. 3, is impossible. Since the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ in $\phi(\Delta_1^{\frac{1}{m}})$ are not all zero, the equation (20) gives us, by §5, $\omega\Delta_1^{\frac{1}{m}} = l_1$, ω being an m^{th} root of unity, and l_1 involving only surds in $\phi(\Delta_1^{\frac{1}{m}})$ exclusive of $\Delta_1^{\frac{1}{m}}$. In l_1 we may conceive $z_2^{\frac{1}{c}}$ changed into $\theta_1 z_1^{\frac{1}{c}}$. Then l_1 involves only surds distinct from $\Delta_1^{\frac{1}{m}}$, all of them except the primitive c^{th} root of unity θ_1 being surds that occur in r_1. This makes the equation $\omega\Delta_1^{\frac{1}{m}} = l_1$ of the inadmissible type (3). Hence the equations $X_1 = 0$ and $X_1' = 0$ have no root in common.

§27. *Cor.* 5. Let X_2 be the continued product of the terms in (16). Then X_2, modified according to §21, is clear of $z_1^{\frac{1}{c}}$, in the same way in which X_1 is clear of $\Delta_1^{\frac{1}{m}}$. Also since, by Cor. 2, each of the equations $X_1 = 0$, $X_1' = 0$, etc., has m unequal particular cognate forms of R for its roots, and since, by Cor. 4, no two of these equations have a root in common, the mc roots of the equation $X_2 = 0$ are unequal particular cognate forms of R.

§28. Proposition VI. Let the simplified expression r_1, modified according to §21, be a root of the rational irreducible equation of the N^{th} degree, $F(x) = 0$. Then if $\Delta_1^{\frac{1}{m}}$, not a root of unity, be a surd of the highest rank in r_1, N is a multiple of m. But if r_1 involve only surds that are roots of unity, one of them being the primitive n^{th} root of unity, N is a multiple of a measure of $n - 1$.

First, let $\Delta_1^{\frac{1}{m}}$, not a root of unity, be a surd of the highest rank in r_1. Taking the expression (1) to be r_1, let X_1 be formed as in §24, and let it be modified according to §21. It is clear of the surd $\Delta_1^{\frac{1}{m}}$. Should it involve a surd that is not a root of unity, let X_2 be formed as in §27. Setting out from r_1 we arrived by one step at X_1, an expression clear of $\Delta_1^{\frac{1}{m}}$, and such that the roots of the equation $X_1 = 0$ are unequal particular cognate forms of R. A second step brought us to X_2, an expression clear of the additional surd $z_1^{\frac{1}{c}}$, and such that the mc roots of the equation $X_2 = 0$ are unequal particular cognate forms of R.

Thus we can go on till, in the series X_1, X_2, etc., we reach a term X_e into which no surds enter that are not roots of unity, the $mc \ldots l$ roots of the equation $X_e = 0$ being unequal particular cognate forms of R. Should X_e, modified according to §21, not be rational, its form, by Prop. IV, putting d for $mc \ldots l$, is

$$X_e = x^d - (p_1 C_1 + \ldots + p_m C_m) x^{d-1} + (q_1 C_1 + \ldots + q_m C_m) x^{d-2} + \text{etc.};$$

where, one of the roots occurring in X_e being the primitive n^{th} root of unity ω_1, the coefficients p_1, q_1, etc., are clear of ω_1; and C_1 is the sum of the cycle of primitive n^{th} roots of unity (8) containing s or $\dfrac{n-1}{m}$ terms; and, the cycle (9) containing all the primitive n^{th} roots of unity, the change of ω_1 into ω_1^β causes C_1 to become C_2, and C_2 to become C_3, and so on, C_m becoming C_1. As was explained at the close of §20, the cycle (8) may be reduced to a single term, which is then identical with C_1. It will also not be forgotten that the roots of unity such as the n^{th} here spoken of are, according to §1, subject to the condition that the numbers such as n are prime. When C_1 in X_e is changed successively into C_1, C_2, etc., let X_e become

$$X_e,\ X_e',\ X_e'', \ldots ,\ X_e^{(m-1)}. \tag{22}$$

If X_{e+1} be the continued product of the terms in (22), the dm roots of the equation $X_{e+1} = 0$ can be shown to be unequal particular cognate forms of R. For, no two terms in (22) as X_e and X_e' are identical; because, if they were, X_e would remain unaltered by the change of ω_1 into ω_1^β; which, by Prop. IV, because ω_1^β is not a term in the cycle (8), is impossible. It follows that no two of the equations $X_e = 0$, $X_e' = 0$, etc., have a root in common. For, if the equations $X_e = 0$ and $X_e' = 0$ had a root in common, since X_e and X_e' are not identical, X_e would have a lower measure involving only surds found in X_e, because the surds in X_e' are the same with those in X_e. Let $\phi(x)$ be this lower measure of X_e, and let r_1 be a root of the equation $\phi(x) = 0$. Then, by Cor. Prop. II, all the d roots of the equation $X_e = 0$ are roots of the equation $\phi(x) = 0$; which is impossible. In the same way it can be proved that no equation in the series $X_e = 0$, $X_e' = 0$, etc., has equal roots. Since no one of these equations has equal roots, and no two of them have a root in common, the dm roots of the equation $X_{e+1} = 0$ are unequal particular cognate forms of R. Also X_{e+1}, modified according to §21, is clear of the primitive n^{th} roots of unity. Should X_{e+1} not be rational, we can deal with it as we did with X_e. Going on in this way, we ultimately reach a *rational* expression X_z such that

the $dm \ldots g$ roots of the equation $X_z = 0$ are unequal particular cognate forms of R. This equation must be identical with the equation $F(x) = 0$ of which r_1 is a root. For, by Prop. III, the equation $F(x) = 0$ has for its roots the unequal particular cognate forms of R. Therefore, because the roots of the equation $X_z = 0$ are all unequal and are at the same time particular cognate forms of R, X_z must be either a lower measure of $F(x)$ or identical with $F(x)$. But $F(x)$, being irreducible, has no lower measure. Therefore X_z is identical with $F(x)$. Therefore, the equation $F(x) = 0$ being of the N^{th} degree, $N = mc \ldots lm \ldots g$. Hence N is a multiple of m. This is the result arrived at when r_1 involves a surd of the highest rank $\Delta_1^{\frac{1}{m}}$ not a root of unity. Should r_1 involve no surds except roots (see §1) of unity, we should then have set out from X_e regarded as identical with $x - r_1$. The result would have been $N = m \ldots g$. Therefore N is a multiple of m; and, because m is here the number of cycles of s terms each, that make up the series of the primitive n^{th} roots of unity, $ms = n - 1$. Therefore N is a multiple of a measure of $n - 1$.

§29. *Cor.* Let N be a prime number. Then, if r_1 involve a surd of the highest rank $\Delta_1^{\frac{1}{m}}$ not a root (see §1) of unity, $N = m$; for, the series of integers m, c, etc., of which N is the continued product, is reduced to its first term. If r_1 involve only surds that are roots of unity, $n - 1$ is a multiple of N; for $N = m \ldots g$; therefore, because N is prime, it is equal to m; but $ms = n - 1$; therefore $n - 1 = sN$.

The Solvable Irreducible Equation of the m^{th} Degree, m Prime.

§30. The principles that have been established may be illustrated by an examination of the solvable irreducible rational equation of the m^{th} degree $F(x) = 0$, m being prime. Two cases may be distinguished, though it will be found that the roots can in the two cases be brought under a common form; the one case being that in which the simplified root r_1 is, and the other that in which it is not, a rational function of roots of unity, that is, according to §1, of roots of unity having the denominators of their indices prime numbers. The equation $F(x) = 0$ may be said to be in the former case *of the first class*, and in the latter *of the second class*.

The Equation $F(x) = 0$ of the First Class.

§31. In this case, by Cor. Prop. VI, r_1 being modified according to §21, if one of the roots involved in r_1 be the primitive n^{th} root of unity ω_1, $n - 1$ is a

multiple of m. Also the expression written X_e in Prop. VI is reduced to $x - r_1$, so that $r_1 = p_1 C_1 + p_2 C_2 + \ldots + p_m C_m$. The m roots of the equation $F(x) = 0$ being r_1, r_2, etc., we must have

$$\left.\begin{array}{l} r_1 = p_1 C_1 + p_2 C_2 + \ \ldots \ + p_m C_m, \\ r_2 = p_m C_1 + p_1 C_2 + \ \ldots \ + p_{m-1} C_m, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ r_m = p_2 C_1 + p_3 C_2 + \ \ldots \ + p_1 C_m. \end{array}\right\} \qquad (23)$$

For, by Prop. II, because r_1 is a root of the equation $F(x) = 0$, all the expressions on the right of the equations (23) are roots of that equation. And no two of these expressions are equal to one another. For, take the first two. If these were equal, we should have $(p_m - p_1)C_1 + (p_1 - p_2)C_2 + \text{etc.} = 0$. Therefore, by §13, each of the terms $p_m - p_1$, $p_1 - p_2$, etc., is zero. This makes p_1, p_2, etc., all equal to one another. Therefore $r_1 = -p_1$; so that the primitive n^{th} root of unity is eliminated from r_1; which, by §21, is impossible. Hence the values of the m roots of the equation $F(x) = 0$ are those given in (23).

§32. Let r'_1 be one of the particular cognate forms of the generic expression R under which the simplified expression r_1 falls. Then, because, by Prop. II, all the particular cognate forms of R are roots of the equation $F(x) = 0$, r'_1 is equal to one of the m terms r_1, r_2, etc., say to r_z. I will now show that the changes of the surds involved that cause r_1 to become r'_1, whose value is r_z, cause r_2 to receive the value r_{z+1}, and r_3 to receive the value r_{z+2}, and so on. This may appear obvious on the face of the equations (23); but, to prevent misunderstanding, the steps of the deduction are given. Any changes made in r_1 must transform C_1 into C_s, one of the m terms C_1, C_2, etc. In passing from r_1 to r'_1, while C_1 becomes C_s, let r_2 become r'_2, and p_1 become p'_1, and p_2 become p'_2, and so on. The change that causes C_1 to become C_s transforms C_2 into C_{s+1}, and C_3 into C_{s+2}, and so on. Therefore, it being understood that p_{m+1}, p_{m+2}, etc., are the same as p_1, p_2, etc., respectively, and in like manner that C_{m+1}, C_{m+2}, etc., are the same as C_1, C_2, etc., respectively,

$$r'_1 = p'_1 C_s + p'_2 C_{s+1} + \text{etc.},$$
$$\text{and } r'_2 = p'_m C_s + p'_1 C_{s+1} + \text{etc.};$$

which may be otherwise written

$$\left.\begin{array}{l} r'_1 = p'_{m+2-s} C_1 + p'_{m+3-s} C_2 + \text{etc.}, \\ r'_2 = p'_{m+1-s} C_1 + p'_{m+2-s} C_2 + \text{etc.} \end{array}\right\} \qquad (24)$$

Therefore, from (24) and (23),

$$C_1(p_{m+2-z}-p'_{m+2-s})+C_2(p_{m+3-z}-p'_{m+3-s})+\text{etc.}=0.$$

Therefore, by §13, $p'_{m+2-s}=p_{m+2-z}$, $p'_{m+3-s}=p_{m+3-z}$, etc. Hence the second of the equations (24) becomes

$$r'_2=p_{m+1-z}C_1+p_{m+2-z}C_2+\text{etc.}=r_{z+1}.$$

Thus r_2 is transformed into r_{z+1}. In like manner r_3 receives the value r_{z+2}, and so on.

§33. By Cor. Prop. VI, the primitive n^{th} root of unity being one of those involved in r_1, $n-1$ is a multiple of m. In like manner, if the primitive a^{th} root of unity be involved in r_1, $a-1$ is a multiple of m, and so on. Therefore, if t_1 be the primitive m^{th} root of unity, t_1 is distinct from all the roots involved in r_1.

§34. From this it follows that, if the circle of roots $r_1, r_2, \ldots, r_m$, be arranged, beginning with r_c, in the order r_c, r_{c+1}, r_{c+2}, etc., and again, beginning with r_s, in the order r_s, r_{s+1}, r_{s+2}, etc., and, if t_1^a being one of the primitive m^{th} roots of unity,

$$r_c+r_{c+1}t_1+r_{c+2}t_1^2+\text{etc.}=r_s+r_{s+1}t_1^a+r_{s+2}t_1^{2a}+\text{etc.},\tag{25}$$

$r_c=r_s$. It is understood that in the series r_c, r_{c+1}, etc., when r_m is reached, the next in order is r_1, so that r_{m+1} is the same as r_1, and so on. In like manner r_{s+1} is the same as r_1, and so on. Since r_1, r_2, etc., do not involve the primitive m^{th} root of unity t_1, we can, by §12, substitute for t_1 in (25) successively the different primitive m^{th} roots of unity. Let this be done. Then, by addition,

$$mr_c-(r_1+r_2+\text{etc.})=mr_s-(r_1+r_2+\text{etc.}).\quad\text{Therefore } r_c=r_s.$$

§35. Proposition VII. Putting

$$\left.\begin{aligned}\Delta_1^{\frac{1}{m}}&=r_1+t_1r_2+t_1^2r_3+\ \ldots\ +t_1^{m-1}r_m,\\ \Delta_2^{\frac{1}{m}}&=r_1+t_1^2r_2+t_1^4r_3+\ \ldots\ +t_1^{2(m-1)}r_m,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \Delta_{m-1}^{\frac{1}{m}}&=r_1+t_1^{-1}r_2+t_1^{-2}r_3+\ \ldots\ +t_1r_m,\end{aligned}\right\}\tag{26}$$

the terms, $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_{m-1}$, (27)

are the roots of a rational irreducible equation of the $(m-1)^{\text{th}}$ degree $\phi(x)=0$, which may be said to be *auxiliary* to the equation $F(x)=0$.

For, let Δ be the generic expression of which Δ_1 is a particular cognate form; and let Δ' denote any one indifferently of the $m-1$ particular cognate forms of Δ in (27). Because, by §33, the primitive m^{th} root of unity does not

enter into r_1, r_2, etc., no changes made in r_1, r_2, etc., affect t_1. Also, by §32, if r_1 becomes r_z, r_2 becomes r_{z+1}, r_3 becomes r_{z+2}, and so on. Therefore the expression
$$(r_z + tr_{z+1} + t^2 r_{z+2} + \text{etc.})^m,$$
contains all the particular cognate forms of Δ; where z may be any number in the series $1, 2, \ldots, m-1$; and t denotes any one indifferently of the primitive m^{th} roots of unity. But this is equal to
$$\{t^{1-z}(r_1 + tr_2 + t^2 r_3 + \text{etc.})\}^m \text{ or } \Delta'.$$
The conclusion established means that all the differences of value that can present themselves in the particular cognate forms of Δ must arise from the different values of t that are taken in Δ', while the expressions r_1, r_2, etc., remain unaltered. And t has not more than $m-1$ values. Hence there are not more than $m-1$ unequal particular cognate forms of Δ. But the $m-1$ forms obtained by taking the different values of t in Δ' are all unequal. For, selecting t_1 and t_1^a, two distinct values of t, suppose if possible that
$$(r_1 + t_1 r_2 + \text{etc.})^m = (r_1 + t_1^a r_2 + \text{etc.})^m \ \therefore\ t_1^s(r_1 + t_1 r_2 + \text{etc.}) = r_1 + t_1^a r_2 + \text{etc.},$$
s being a whole number. This may be written
$$r_{m+1-s} + r_{m+2-s}t_1 + \text{etc.} = r_1 + t_1^a r_2 + \text{etc.} \tag{28}$$
Therefore, by §34, $r_{m+1-s} = r_1$. This means, since all the m terms r_1, r_2, etc., are unequal, that $s = 0$. Hence (28) becomes $r_1 + r_2 t_1 + \text{etc.} = r_1 + r_2 t_1^a + \text{etc.}$ Therefore
$$r_2 + r_3 t_1^a + \text{etc.} = r_2 t_1^{1-a} + r_3 t_1^{2-a} + \text{etc.} = r_{a+1} + r_{a+2}t_1 + \text{etc.}$$
Therefore, by §35, $r_2 = r_{a+1}$. Therefore, because all the m terms r_1, r_2, etc., are unequal, $a = 1$; which, because t_1 and t_1^a were supposed to be distinct primitive m^{th} roots of unity, is impossible. Therefore no two of the terms in (27) are equal to one another. And it has been proved that there is no particular cognate form of Δ which is not equal to a term in (27). Therefore the terms in (27) are the unequal particular cognate forms of Δ. Therefore, by Prop. III, they are the roots of a rational irreducible equation.

§36. PROPOSITION VIII. The roots of the equation $\phi(x) = 0$ auxiliary (see §35) to $F(x) = 0$ are rational functions of the primitive m^{th} root of unity.

For, let the value of Δ_1, obtained from (26), and modified according to §21, be
$$\Delta_1 = k_1 + k_2 t_1 + k_3 t_1^2 + \ldots + k_m t_1^{m-1},$$
where k_1, k_2, etc., are clear of t_1. Suppose if possible that k_1, k_2, etc., are not rational. We may take the primitive n^{th} root of unity ω_1 to be present in these coefficients. But ω_1 occurs in r_1, r_2, etc., and therefore also in Δ_1, only in the

expressions C_1, C_2, etc. Therefore $\Delta_1 = d_1C_1 + \ldots + d_mC_m$; where d_1, etc., are clear of ω_1. The coefficients d_1, d_2, etc., cannot all be equal; for this would make $\Delta_1 = -d_1$; which, by §21, is impossible. Hence m unequal values of the generic expression Δ are obtained by changing C_1 successively into C_1, C_2, etc., namely,

$$\begin{gathered} d_1C_1 + d_2C_2 + \ \ldots \ + d_mC_m, \\ d_mC_1 + d_1C_2 + \ \ldots \ + d_{m-1}C_m, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ d_2C_1 + d_3C_2 + \ \ldots\ldots \ + d_1C_m. \end{gathered}$$

To show that these expressions are all unequal, take the first two. If these were equal, we should have

$$(d_m - d_1)C_1 + (d_1 - d_2)C_2 + \text{etc.} = 0.$$

Therefore, by §13, $d_m - d_1 = 0$, $d_1 - d_2 = 0$, and so on; which, because d_1, d_2, etc., are not all equal to one another, is impossible. Since then Δ has at least m unequal particular cognate forms, Δ_1 is, by Prop. III, the root of a rational irreducible equation of a degree not lower than the m^{th}; which, by Prop. VII, is impossible. Therefore k_1, k_2, etc., are rational. Hence each of the expressions in (27) is a rational function of t_1.

§37. *Cor.* Any expression of the type $k_1 + k_2t_1 + k_3t_1^2 +$ etc., which is such that all the unequal particular cognate forms of the generic expression under which it falls are obtained by substituting for t_1 successively the different primitive m^{th} roots of unity, while k_1, k_2, etc., remain unaltered, is a rational function of t_1. For, in the Proposition, Δ_1 or $k_1 + k_2t_1 +$ etc. was shown to be a rational function of t_1, the conclusion being based on the circumstance that Δ_1 satisfies the condition specified.

§38. Proposition IX. If g be the sum of the roots of the equation $F(x) = 0$,

$$r_1 = \tfrac{1}{m}\left(g + \Delta_1^{\frac{1}{m}} + a_1\Delta_1^{\frac{2}{m}} + b_1\Delta_1^{\frac{3}{m}} + \ldots + e_1\Delta_1^{\frac{m-2}{m}} + h_1\Delta_1^{\frac{m-1}{m}}\right); \tag{29}$$

where a_1, b_1, etc., involve no surd that is not subordinate (see §3) to $\Delta_1^{\frac{1}{m}}$.

For, z being one of the whole numbers $1, 2, \ldots, m-1$, put

$$p_z = (r_1 + t_1^z r_2 + t_1^{2z} r_3 + \text{etc.})(r_1 + t_1r_2 + t_1^2r_3 + \text{etc.})^{-z}. \tag{30}$$

Multiply the first of its factors by t_1^{-z} and the second by t_1^z. Then

$$p_z = (r_2 + t_1^z r_3 + t_1^{2z} r_4 + \text{etc.})(r_2 + t_1r_3 + t_1^2r_4 + \text{etc.})^{-z}. \tag{31}$$

Hence p_z does not alter its value when we change r_1 into r_2, r_2 into r_3, and so on. In like manner it does not alter its value when we change r_1 into r_a, r_2 into r_{a+1}, and so on. Therefore, by §33, p_z is not changed by any alterations that may be made in r_1, r_2, etc., while t_1 remains unaltered. Consequently, if p_z be

a particular cognate form of P, all the unequal particular cognate forms of P are obtained by substituting for t_1 successively in p_z the different primitive m^{th} roots of unity, while r_1, r_2, etc., remain unaltered. Therefore, by Cor., Prop. VIII, p_z is a rational function of t_1. When $z = 2$, let $p_z = a_1$; when $z = 3$, let $p_z = b_1$, and so on. Then, from (26) and (30), $\Delta_2^{\frac{1}{m}} = a_1 \Delta_1^{\frac{2}{m}}$, $\Delta_3^{\frac{1}{m}} = b_1 \Delta_1^{\frac{3}{m}}$, and so on. But, from (27), since g is the sum of the roots of the equation $F(x) = 0$,

$$r_1 = \frac{1}{m}(g + \Delta_1^{\frac{1}{m}} + \Delta_2^{\frac{1}{m}} + \ldots + \Delta_{m-1}^{\frac{1}{m}}).$$

By putting $a_1 \Delta_1^{\frac{2}{m}}$ for $\Delta_2^{\frac{1}{m}}$, $b_1 \Delta_1^{\frac{3}{m}}$ for $\Delta_3^{\frac{1}{m}}$, and so on, this becomes (29). Because a_1, b_1, etc., are rational functions of t_1, while Δ_1, the root of a rational irreducible equation of the $(m-1)^{\text{th}}$ degree, is also a rational function of t_1, the coefficients a_1, b_1, etc., involve no surd that is not subordinate to $\Delta_1^{\frac{1}{m}}$.

§39. PROPOSITION X. If the prime number m be odd, the expressions

$$\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}},\ \Delta_2^{\frac{1}{m}} \Delta_{m-2}^{\frac{1}{m}},\ \ldots,\ \Delta_{\frac{m-1}{2}}^{\frac{1}{m}} \Delta_{\frac{m+2}{m}}^{\frac{1}{m}}, \tag{32}$$

are the roots of a rational equation of the $\left(\frac{m-1}{2}\right)^{\text{th}}$ degree.

By §32, when r_1 is changed into r_z, r_2 becomes r_{z+1}, r_3 becomes r_{z+2}, and so on. Hence the terms $r_1 r_2$, $r_2 r_3$, $\ldots$, $r_m r_1$, form a cycle, the sum of the terms in which may be denoted by the symbol Σ_2^1. In like manner the sum of the terms in the cycle $r_1 r_3$, $r_2 r_4$, $\ldots$, $r_m r_2$, may be written Σ_3^1. And so on. In harmony with this notation, the sum of the m terms r_1^2, r_2^2, etc., may be written Σ_1^1. Now r_1 can only be changed into one of the terms r_1, r_2, etc.; and we have seen that, when it becomes r_z, r_2 becomes r_{z+1}, and so on. Such changes leave the cycle $r_1 r_2$, $r_2 r_3$, etc., as a whole unaltered. Therefore, by Prop. III, Σ_2^1 is the root of a simple equation, or has a rational value. In like manner each of the expressions

$$\Sigma_1^1,\ \Sigma_2^1,\ \Sigma_3^1,\ \ldots,\ \Sigma_m^1, \tag{33}$$

has a rational value. From (26), by actual multiplication,

$$\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}} = \Sigma_1^1 + (\Sigma_2^1) t_1 + (\Sigma_3^1) t_1^2 + \text{etc.}$$

But Σ_2^1, Σ_3^1, etc., are respectively identical with Σ_m^1, Σ_{m-1}^1, etc. Therefore

$$\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}} = \Sigma_1^1 + (\Sigma_2^1)(t_1 + t_1^{-1}) + (\Sigma_3^1)(t_1^2 + t_1^{-2}) + \text{etc.} \tag{34}$$

Hence, since the terms in (33) are all rational, and since the terms in (32) are respectively what $\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}}$ becomes by changing t_1 successively into the $\frac{m-1}{2}$

terms t_1, t_1^2, etc., the terms in (32) are the roots of a rational equation of the $\left(\frac{m-1}{2}\right)^{\text{th}}$ degree.

§40. For the solution of the equation $x^n - 1 = 0$, n being a prime number such that m is a prime measure of $n-1$, it is necessary to obtain the solution of the equation of the m^{th} degree which has for one of its roots the sum of the $\frac{n-1}{m}$ terms in a cycle of primitive n^{th} roots of unity. This latter equation will be referred to as the *reducing Gaussian equation* of the m^{th} degree to the equation $x^n - 1 = 0$.

§41. Proposition XI. When the equation $F(x) = 0$ is the reducing Gaussian (see §40) of the m^{th} degree to the equation $x^n - 1 = 0$, each of the $\frac{m-1}{2}$ expressions in (32) is equal to n.

Let the sum of the primitive n^{th} roots of unity forming the cycle (8), which sum has in preceding sections been indicated by the symbol C_1, be the root r_1 of the equation $F(x) = 0$. This implies, since s is the number of the terms in (8), that $ms = n - 1$. Let us reason first on the assumption that the cycle (8) is made up of pairs of reciprocal roots ω_1 and ω_1^{-1}, and so on. Then, because the cycle consists of $\frac{s}{2}$ pairs of reciprocal roots, C_1^2 or r_1^2 is the sum of s^2 terms, each an n^{th} root of unity. Among these unity occurs s times. Let ω_1 occur h_1 times; and let ω_1^λ, the second term in (8), occur h' times. Since ω_1^λ may be made the first term in the cycle (8), it must, under the new arrangement, present itself in the value of r_1^2, precisely where ω_1 previously appeared. That is to say, $h' = h_1$. In like manner each of the terms in (8) occurs exactly h_1 times in the expression for r_1^2. The cycle (9) being that which contains all the primitive n^{th} roots of unity, let us, adhering to the notation of previous sections, suppose that, when ω_1 is changed into ω_1^β, C_1 or r_1 becomes C_2 or r_2, C_2 or r_2 becomes C_3 or r_3, and so on. On the same grounds on which every term in (8) occurs the same number of times in the value of r_1^2, each term in the cycle of terms whose sum is C_2 occurs the same number of times; and so on. Therefore

$$
\begin{aligned}
r_1^2 &= s + h_1 C_1 + h_2 C_2 + \ldots + h_m C_m. \\
r_2^2 &= s + h_m C_1 + h_1 C_2 + \ldots + h_{m-1} C_m, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
r_m^2 &= s + h_2 C_1 + h_3 C_2 + \ldots + h_1 C_m.
\end{aligned}
$$

Therefore, keeping in view (11), $\Sigma_1^1 = ms - (h_1 + h_2 + \ldots + h_m)$. But $s^2 - s$ is the number of the terms in the value of r_1^2 which are primitive n^{th} roots of unity. And this must be equal to $s(h_1 + \ldots + h_m)$. Therefore

$$h_1 + h_2 + \ldots + h_m = s - 1 \quad \therefore \ \Sigma_1^1 = ms + 1 - s = n - s.$$

Again, because r_1 is made up of pairs of reciprocal roots, and because therefore unity does not occur among the s^2 terms of which $r_1 r_2$ is the sum,

$$\begin{aligned} r_1 r_2 &= k_1 C_1 + k_2 C_2 + \ldots + k_m C_m, \\ r_2 r_3 &= k_m C_1 + k_1 C_2 + \ldots + k_{m-1} C_m, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ r_m r_1 &= k_2 C_1 + k_3 C_2 + \ldots + k_1 C_m; \end{aligned}$$

where k_1, k_2, etc., are whole numbers whose sum is s. Therefore $\Sigma_2^1 = -s$. In like manner each of the terms in (33) except the first is equal to $-s$. Therefore (34) becomes $\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}} = (n - s) - s(t_1 + t_1^2 + \text{etc.}) = n$. Let us reason now on the assumption that the cycle (8) is not made up of pairs of reciprocal roots. It contains in that case no reciprocal roots. By the same reasoning as above we get $\Sigma_1^1 = -s$. As regards the terms in (33) after the first, one of the terms C_1, C_2, etc., say C_z, must be such that the n^{th} roots of unity of which it is the sum are reciprocals of those of which C_1 is the sum. In passing from C_1 to C_z, we change r_1 into r_z. In fact, C_1 being r_1, C_z is r_z. This being kept in view, we get, by the same reasoning as above, $\Sigma_z^1 = n - s$. But, if any of the expressions C_1, C_2, etc., except C_z, be selected, say C_a, none of the roots in (8) are reciprocals of any of those of which C_a is the sum. Therefore $\Sigma_a^1 = -s$. Therefore, from (34)

$$\Delta_1^{\frac{1}{m}} \Delta_{m-1}^{\frac{1}{m}} = -s + (n - s) t_1^{z-1} - s\{(t_1 + t_1^2 + \ldots + t_1^{m-1}) - t_1^{z-1}\} = n.$$

In like manner every one of the expressions in (34) can be shown to have the value n.

§42. Two numerical illustrations of the law established in the preceding section may be given. The reducing Gaussian equation of the third degree to the equation $x^{19} - 1 = 0$ is $x^3 - x^2 - 6x - 7 = 0$; which gives

$$r_1 = \tfrac{1}{3}(-1 + \Delta_1^{\frac{1}{3}} + \Delta_2^{\frac{1}{3}}),\ 2\Delta_1 = 19(7 + 3\sqrt{3}),\ 2\Delta_2 = 19(7 - 3\sqrt{3}),\ \Delta_1^{\frac{1}{3}} \Delta_2^{\frac{1}{3}} = 19.$$

The next example is taken from Lagrange's Theory of Algebraical Equations, Note XIV, §30. The Gaussian of the fifth degree to the equation $x^{11} - 1 = 0$ is $x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = 0$; which gives

$$r_1 = \tfrac{1}{5}(-1 + \Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}} + \Delta_3^{\frac{1}{5}} + \Delta_4^{\frac{1}{5}});$$

$4\Delta_1 = 11(-89 - 25\sqrt{5} + 5p - 45q)$, $4\Delta_2 = 11(-89 + 25\sqrt{5} - 45p - 5q)$,
$4\Delta_4 = 11(-89 - 25\sqrt{5} - 5p + 45q)$, $4\Delta_3 = 11(-89 + 25\sqrt{5} + 45p + 5q)$,
$p = \sqrt{(-5 - 2\sqrt{5})}$, $q = \sqrt{(-5 + 2\sqrt{5})}$, $pq = -\sqrt{5}$ $\therefore \Delta_1\Delta_4 = 11^5$.

§43. Proposition XII. To solve the Gaussian.

The path we have been following leads directly, assuming the primitive m^{th} root of unity t_1 to be known, to the solution of the reducing Gaussian equation of the m^{th} degree to the equation $x^n - 1 = 0$. For, as in §41, the roots of the Gaussian are C_1, C_2, etc. Therefore g, the sum of the roots, is -1. Therefore

$$r_1 = \tfrac{1}{m}(-1 + \Delta_1^{\frac{1}{m}} + \Delta_2^{\frac{1}{m}} + \ldots + \Delta_{m-1}^{\frac{1}{m}}). \tag{35}$$

By Prop. VIII, Δ_1, Δ_2, etc., are rational functions of t_1. Therefore

$$\left.\begin{array}{l} \Delta_1 = k_1 + k_2 t_1 + k_3 t_1^2 + \ldots + k_m t_1^{m-1} \\ \Delta_2 = k_1 + k_2 t_1^2 + k_3 t_1^4 + \ldots + k_m t_1^{2(m-1)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_{m-1} = k_1 + k_2 t_1^{-1} + k_3 t_1^{-2} + \ldots + k_m t_1; \end{array}\right\} \tag{36}$$

where k_1, k_2, etc., are rational. From the first of equations (26), putting C_1 for r_1, C_2 for r_2, and so on, $\Delta_1 = (C_1 + t_1 C_2 + \text{etc.})^m$.
By actual involution this gives us k_1, k_2, etc., as determinate functions of C_1, C_2, etc., and therefore as known rational quantities. For instance take k_1. Being a determinate function of C_1, C_2, etc., we have

$$k_1 = q_1 + q_2 C_1 + q_3 C_2 + \ldots + q_m C_{m-1};$$

where q_1, q_2, etc., are known rational quantities. But, by §13, the rational coefficients $q_1 - k_1$, q_2, etc., are all equal to one another. Therefore $k_1 = q_1 - q_2$. In like manner k_2, k_3, etc., are known. Therefore, from (36), Δ_1, Δ_2, etc., are known. Therefore, from (35), r_1 is known.

§44. Proposition XIII. The law established in Prop. X falls under the following more general law. The $m-1$ expressions in each of the groups

$$\left.\begin{array}{l} (\Delta_1^{\frac{1}{m}}\Delta_{m-1}^{\frac{1}{m}},\ \Delta_2^{\frac{1}{m}}\Delta_{m-2}^{\frac{1}{m}}, \ldots,\ \Delta_{m-1}^{\frac{1}{m}}\Delta_1^{\frac{1}{m}},) \\ (\Delta_1^{\frac{2}{m}}\Delta_{m-2}^{\frac{1}{m}},\ \Delta_2^{\frac{2}{m}}\Delta_{m-4}^{\frac{1}{m}}, \ldots,\ \Delta_{m-1}^{\frac{2}{m}}\Delta_2^{\frac{1}{m}},) \\ (\Delta_1^{\frac{3}{m}}\Delta_{m-3}^{\frac{1}{m}},\ \Delta_2^{\frac{3}{m}}\Delta_{m-6}^{\frac{1}{m}}, \ldots,\ \Delta_{m-1}^{\frac{3}{m}}\Delta_3^{\frac{1}{m}},) \end{array}\right\} \tag{37}$$

and so on, are the roots of a rational equation of the $(m-1)^{\text{th}}$ degree.

The $m-1$ terms in the first of the groups (37) are the $\dfrac{m-1}{2}$ terms in (32) each taken twice. Therefore, by Prop. X, the law enunciated in the present Proposition is established so far as this group is concerned. The general proof

is as follows. By (30) in §38, taken in connection with (26), $p_{m-z}\Delta_1^{\frac{m-z}{m}} = \Delta_{m-z}^{\frac{1}{m}}$. Therefore $\Delta_1^{\frac{z}{m}}\Delta_{m-z}^{\frac{1}{m}} = p_{m-z}\Delta_1$. But, by §38, p_{m-z} is a rational function of t_1; and, by Prop. VIII, Δ_1 is a rational function of t_1. Therefore $\Delta_1^{\frac{z}{m}}\Delta_{m-z}^{\frac{1}{m}}$ is a rational function of t_1. Also from the manner in which p_{m-z} is formed, when t_1 in $p_{m-z}\Delta_1$ is changed successively into $t_1, t_1^2, \ldots, t_1^{m-1}$, the expression $\Delta_1^{\frac{z}{m}}\Delta_{m-z}^{\frac{1}{m}}$ is changed successively into the $m-1$ terms of that one of the groups (37) whose first term is $\Delta_1^{\frac{z}{m}}\Delta_{m-z}^{\frac{1}{m}}$. Therefore the terms in that group are the roots of a rational equation.

§45. *Cor.* The law established in the Proposition may be brought under a yet wider generalization. The expression

$$\Delta_1^{\frac{a}{m}}\Delta_2^{\frac{b}{m}}\Delta_3^{\frac{c}{m}}\ldots\Delta_{m-1}^{\frac{s}{m}} \tag{38}$$

is the root of a rational equation of the $(m-1)^{\text{th}}$ degree, if

$$a + 2b + 3c + \ldots + (m-1)s = Wm,$$

W being a whole number. For, by (30) in connection with (26), $\Delta_2^{\frac{1}{m}} = p_2\Delta_1^{\frac{2}{m}}$, $\Delta_3^{\frac{1}{m}} = p_3\Delta_1^{\frac{3}{m}}$, and so on. Therefore (38) has the value

$$(p_2^b p_3^c \ldots)\Delta_1^{\frac{a+2b+3c+\ldots+(m-1)s}{m}}, \text{ or } (p_2^b p_3^c \ldots)\Delta_1^W.$$

This is a rational function of t_1, and therefore the root of a rational equation of the $(m-1)^{\text{th}}$ degree.

THE EQUATION $F(x)=0$ OF THE SECOND CLASS.

§46. We now suppose that the simplified root r_1 of the rational irreducible equation $F(x)=0$ of the m^{th} degree, m prime, involves, when modified according to §21, a principal surd not a root of unity. It must not be forgotten that, when we thus speak of roots of unity, we mean, according to §1, roots which have prime numbers for the denominators of their indices. In this case conclusions can be established similar to those reached in the case that has been considered. The root r_1 is still of the form (29). The equation $F(x)=0$ has still an auxiliary of the $(m-1)^{\text{th}}$ degree, whose roots are the m^{th} powers of the expressions

$$\Delta_1^{\frac{1}{m}},\ a_1\Delta_1^{\frac{2}{m}},\ b_1\Delta_1^{\frac{3}{m}},\ \ldots,\ e_1\Delta_1^{\frac{m-2}{m}},\ h_1\Delta_1^{\frac{m-1}{m}}, \tag{39}$$

though the auxiliary here is not necessarily irreducible. Also, substituting the expressions in (39) for $\Delta_1^{\frac{1}{m}}$, $\Delta_2^{\frac{1}{m}}$, etc., in (37), the law of Proposition XIII still holds, together with the corollary in §45.

§47. By Cor. Prop. VI, the denominator of the index of a surd of the highest rank in r_1 is m. Let $\Delta_1^{\frac{1}{m}}$ be such a surd. By §21, the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ in r_1 cannot be all zero. We may take the coefficient of the first power to be distinct from zero and to be $\frac{1}{m}$, for, if it were $\frac{k_1}{m}$, we might substitute $s^{\frac{1}{m}}$ for $k_1\Delta_1^{\frac{1}{m}}$, and so eliminate $\Delta_1^{\frac{1}{m}}$ from r_1, introducing in its room the new surd $s^{\frac{1}{m}}$ with $\frac{1}{m}$ for the coefficient of its first power. We may then put

$$r_1 = \tfrac{1}{m}(g + \Delta_1^{\frac{1}{m}} + a_1\Delta_1^{\frac{2}{m}} + \ldots + e_1\Delta_1^{\frac{m-2}{m}} + h_1\Delta_1^{\frac{m-1}{m}}); \tag{40}$$

where g, a_1, etc., are clear of $\Delta_1^{\frac{1}{m}}$. When $\Delta_1^{\frac{1}{m}}$ is changed successively into $\Delta_1^{\frac{1}{m}}$, $t_1^{-1}\Delta_1^{\frac{1}{m}}$, $t_1^{-2}\Delta_1^{\frac{1}{m}}$, etc., let

$$r_1, r_2, \ldots, r_m, \tag{41}$$

be respectively what r_1 becomes, t_1 being a primitive m^{th} root of unity. By Prop. VI, the terms in (41) are the roots of the equation $F(x) = 0$. Taking r_n, any one of the particular cognate forms of R, let $\Delta_n^{\frac{1}{m}}$, a_n, etc., be respectively what $\Delta_1^{\frac{1}{m}}$, a_1, etc., become in passing from r_1 to r_n; and, when $\Delta_n^{\frac{1}{m}}$ is changed successively into the different m^{th} roots of the determinate base Δ_n, let r_n become

$$r_n, r'_n, r''_n, \ldots, r_n^{(m-1)}. \tag{42}$$

By Prop. II, the terms in (42) are roots of the equation $F(x) = 0$; and, by §23, they are all unequal. Therefore they are identical, in some order, with the terms in (41). Also, the sum of the terms in (41) is g. Therefore g is rational.

§48. Proposition XIV. In r_1, as expressed in (40), $\Delta_1^{\frac{1}{m}}$ is the only principal (see §2) surd.

Suppose, if possible, that there is in r_1 a principal surd $z_1^{\frac{1}{c}}$ distinct from $\Delta_1^{\frac{1}{m}}$. And first, let $z_1^{\frac{1}{c}}$ be not a root of unity. (It will be kept in view that when, in such a case, we speak of roots of unity, the denominators of their indices are understood, according to §1, to be prime numbers.) When $z_1^{\frac{1}{c}}$ is changed into $z_2^{\frac{1}{c}}$, one of the other c^{th} roots of z_1, let r_1, a_1, etc., become respectively r'_1, a'_1, etc. Then

$$mr'_1 = g + \Delta_1^{\frac{1}{m}} + a'_1\Delta_1^{\frac{2}{m}} + \text{etc.} \tag{43}$$

By Prop. II, r'_1 is equal to a term in (41), say to r_n. And, by §48, putting t_{n-1} for t_1^{1-n},

$$mr_n = g + t_{n-1}\Delta_1^{\frac{1}{m}} + t_{n-1}^2 a_1\Delta_1^{\frac{2}{m}} + \text{etc.} \tag{44}$$

Therefore,

$$\Delta_1^{\frac{1}{m}}(1 - t_{n-1}) + \Delta_1^{\frac{2}{m}}(a'_1 - a_1 t_{n-1}^2) + \text{etc.} = 0. \tag{45}$$

This equation involves no surds except those found in the simplified expression r_1, together with the primitive m^{th} root of unity. Therefore the expression on

the left of (45) is in a simple state. Therefore, by §8, the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ are separately zero. Therefore $t_{n-1}=1$, $a'_1=a$, $b'_1=b_1$, and so on. But, as was shown in Prop. V, $z_1^{\frac{1}{c}}$ being a principal surd not a root of unity in the simplified expression a_1, a_1 cannot be equal to a'_1 unless $z_1^{\frac{1}{c}}$ can be eliminated from a_1 without the introduction of any new surd. In like manner b_1 cannot be equal to b'_1 unless $z_1^{\frac{1}{c}}$ can be eliminated from b_1. And so on. Therefore, because $a_1=a'_1$, and $b_1=b'_1$, and so on, $z_1^{\frac{1}{c}}$ admits of being eliminated from r_1 without the introduction of any new surd, which, by §21, is impossible. Next, let $z_1^{\frac{1}{c}}$ be a root (see §1) of unity, which may be otherwise written θ_1. Let the different primitive c^{th} roots of unity be θ_1, θ_2, etc.; and, when θ_1 is changed successively into θ_1, θ_2, etc., let r_1 become successively r_1, r'_1, etc. Suppose if possible that the $c-1$ terms r_1, r'_1, etc., are all equal. Since $z_1^{\frac{1}{c}}$ is a principal surd in r_1, we may put $r_1=h\theta_1^{c-1}+k\theta_1^{c-2}+\ldots+l$; where h, k, etc., are clear of θ_1. Therefore $(c-1)r_1=cl-(h+k+\text{etc.})$ Thus $z_1^{\frac{1}{c}}$ may be eliminated from r_1 without the introduction of any new surd; which by §21 is impossible. Since then the terms r_1, r'_1, etc., are not all equal, let r_1 and r'_1 be unequal. Then r'_1 is equal to a term in (41) distinct from r_1, say to r_n. Expressing mr'_1 and mr_n as in (43) and (44), we deduce (45); which, as above, is impossible.

§49. PROPOSITION XV. Taking r_1, r_n, $\Delta_n^{\frac{1}{m}}$, etc., as in §47, an equation

$$t\Delta_n^{\frac{1}{m}}=p\Delta_1^{\frac{c}{m}} \tag{46}$$

can be formed; where t is an m^{th} root of unity, and c is a whole number less than m but not zero, and p involves only surds subordinate (see §3) to $\Delta_1^{\frac{1}{m}}$ or $\Delta_n^{\frac{1}{m}}$.

By §47, one of the terms in (42) is equal to r_1. For our argument it is immaterial which be selected. Let $r_n=r_1$. Therefore

$$(h_n\Delta_n^{\frac{m-1}{m}}+e_n\Delta_n^{\frac{m-2}{m}}+\ldots+\Delta_n^{\frac{1}{m}})-(h_1\Delta_1^{\frac{m-1}{m}}+e_1\Delta_1^{\frac{m-2}{m}}+\ldots+\Delta_1^{\frac{1}{m}})=0. \tag{47}$$

The coefficients of the different powers of $\Delta_n^{\frac{1}{m}}$ here are not all zero, for the coefficient of the first power is unity. Therefore by §5, an equation $t\Delta_n^{\frac{1}{m}}=l_1$ subsists, t being an m^{th} root of unity, and l_1 involving only surds exclusive of $\Delta_n^{\frac{1}{m}}$ that occur in (47). By Prop. XIV, $\Delta_1^{\frac{1}{m}}$ is a surd of a higher rank (see §3) than any surd in (47) except $\Delta_n^{\frac{1}{m}}$. Therefore we may put

$$l_1=d+d_1\Delta_1^{\frac{1}{m}}+d_2\Delta_1^{\frac{2}{m}}+\ldots+d_{m-1}\Delta_1^{\frac{m-1}{m}};$$

where d, d_1, etc., involve only surds lower in rank than $\Delta_1^{\frac{1}{m}}$. Then

$$\Delta_n = l_1^m = (d + d_1\Delta_1^{\frac{1}{m}} + \text{etc.})^m = d' + d'_1\Delta_1^{\frac{1}{m}} + d'_2\Delta_1^{\frac{2}{m}} + \text{etc.};$$

where d', d'_1, etc., involve only surds lower in rank than $\Delta_1^{\frac{1}{m}}$. By §8, since $\Delta_1^{\frac{1}{m}}$ is a surd in the simplified expression r_1, the coefficients $d' - \Delta_n$, d'_1, etc., in the equation

$$(d' - \Delta_n) + d'_1\Delta_1^{\frac{1}{m}} + d'_2\Delta_1^{\frac{2}{m}} + \text{etc.} = 0 \tag{48}$$

are separately zero. Therefore $(d + d_1\Delta_1^{\frac{1}{m}} + \text{etc.})^m = d'$. And, t_1 being a primitive m^{th} root of unity,

$$(d + d_1 t_1\Delta_1^{\frac{1}{m}} + \text{etc.})^m = d' + d'_1 t_1\Delta_1^{\frac{1}{m}} + \text{etc.} = d'.$$

Therefore, $\quad (d + d_1 t_1 \Delta_1^{\frac{1}{m}} + \text{etc.}) = t_1^a(d + d_1\Delta_1^{\frac{1}{m}} + d_2\Delta_1^{\frac{2}{m}} + \text{etc.}),$

t_1^a being one of the m^{th} roots of unity. In the same way in which the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ in (48) are separately zero, each of the expressions $d(1 - t_1^a)$, $d_1(t_1 - t_1^a)$, etc., must be zero. But not more than one of the $m-1$ factors, $t_1 - t_1^a$, $t_1^2 - t_1^a$, etc., can be zero. Therefore not more than one of the $m-1$ terms, d_1, d_2, etc., is distinct from zero. Suppose if possible that all these terms are zero. Then $t\Delta_n^{\frac{1}{m}} = d$. Therefore the different powers of $\Delta_n^{\frac{1}{m}}$ can be expressed in terms of the surds involved in d and of the m^{th} root of unity. Substitute for $\Delta_n^{\frac{1}{m}}$, $\Delta_n^{\frac{2}{m}}$, etc., in (47), their values thus obtained. Then (47) becomes

$$Q - (h_1\Delta_1^{\frac{m-1}{m}} + \ldots + \Delta_1^{\frac{1}{m}}) = 0; \tag{49}$$

where Q involves no surds, distinct from the primitive m^{th} root of unity, that are not lower in rank than $\Delta_1^{\frac{1}{m}}$; which, because the coefficient of the first power of $\Delta_1^{\frac{1}{m}}$ in (49) is not zero, is, by §8, impossible. Hence there must be one, while at the same time there can be only one of the $m-1$ terms, d_1, d_2, etc., distinct from zero. Let d_c be the term that is not zero. Then $t_1^c - t_1^a = 0$. Therefore $1 - t_1^a$ is not zero. Therefore $d = 0$. Therefore, putting p for d_c, $t\Delta_n^{\frac{1}{m}} = p\Delta_1^{\frac{c}{m}}$.

§50. *Cor.* By the proposition, values of the different powers of $\Delta_n^{\frac{1}{m}}$ can be obtained as follows:

$$t\Delta_n^{\frac{1}{m}} = p\Delta_1^{\frac{c}{m}},\ t^2\Delta_n^{\frac{2}{m}} = q\Delta_1^{\frac{s}{m}},\ t^3\Delta_n^{\frac{3}{m}} = k\Delta_1^{\frac{z}{m}},\ \text{etc.}; \tag{50}$$

where p, q, etc., involve only surds that occur in Δ_1 or Δ_n; and c, s, z, etc., are whole numbers in the series $1, 2, \ldots, m-1$. No two of the numbers c, s, etc., can be the same; for they are the products, with multiples of the prime number m left out, of the terms in the series $1, 2, \ldots, m-1$, by the whole number c which is less than m. Therefore the series c, s, z, etc., is the series $1, 2, \ldots, m-1$, in a certain order.

§51. PROPOSITION XVI. If r_n be one of the particular cognate forms of R, the expressions
$$t\Delta_n^{\frac{1}{m}},\ t^2 a_n \Delta_n^{\frac{2}{m}},\ \ldots,\ t^{m-2} e_n \Delta_n^{\frac{m-2}{m}},\ t^{m-1} h_n \Delta_n^{\frac{m-1}{m}}, \tag{51}$$
are severally equal, in some order, to those in (39), t being one of the m^{th} roots of unity.

By §47, one of the terms in (42) is equal to r_1. For our argument it is immaterial which be chosen. Let $r_n = r_1$. By Cor. Prop. XV, the equations (50) subsist. Substitute in (47) the values of the different powers of $\Delta_n^{\frac{1}{m}}$ so obtained. Then
$$(t^{-1} p \Delta_1^{\frac{c}{m}} + t^{-2} q a_n \Delta_1^{\frac{s}{m}} + \text{etc.}) - (\Delta_1^{\frac{1}{m}} + a_1 \Delta_1^{\frac{2}{m}} + \text{etc.}) = 0. \tag{52}$$
By Cor. Prop. XV, the series $\Delta_1^{\frac{c}{m}}$, $\Delta_1^{\frac{s}{m}}$, etc., is identical, in some order, with the series $\Delta_1^{\frac{1}{m}}$, $\Delta_1^{\frac{2}{m}}$, etc. Also, by §8, since $\Delta_1^{\frac{1}{m}}$ is a surd occurring in the simplified expression r_1, and since besides $\Delta_1^{\frac{1}{m}}$ there are in (52) no surds, distinct from the primitive m^{th} root of unity, that are not lower in rank than $\Delta_1^{\frac{1}{m}}$, if the equation (52) were arranged according to the powers of $\Delta_1^{\frac{1}{m}}$ lower than the m^{th}, the coefficients of the different powers of $\Delta_1^{\frac{1}{m}}$ would be separately zero. Hence $\Delta_1^{\frac{1}{m}}$ is equal to that one of the expressions,
$$t^{-1} p \Delta_1^{\frac{c}{m}},\ t^{-2} q a_n \Delta_1^{\frac{s}{m}},\ \text{etc.} \tag{53}$$
in which $\Delta_1^{\frac{1}{m}}$ is a factor. In like manner $a_1 \Delta_1^{\frac{2}{m}}$ is equal to that one of the expressions (53) in which $\Delta_1^{\frac{2}{m}}$ is a factor. And so on. Therefore the terms $\Delta_1^{\frac{1}{m}}$, $a_1 \Delta_1^{\frac{2}{m}}$, etc., forming the series (39), are severally equal, in some order, to the terms in (53), which are those forming the series (51).

§52. PROPOSITION XVII. The equation $F(x) = 0$ has a rational *auxiliary* (compare Prop. VII) equation $\phi(x) = 0$, whose roots are the m^{th} powers of the terms in (39).

Let the unequal particular cognate forms of the generic expression Δ under which the simplified expression Δ_1 falls be
$$\Delta_1,\ \Delta_2,\ \ldots,\ \Delta_c. \tag{54}$$
By Prop. XVI, there is a value t of the m^{th} root of unity for which the expressions
$$t\Delta_2^{\frac{1}{m}},\ t^2 a_2 \Delta_2^{\frac{2}{m}},\ \ldots,\ t^{m-2} e_2 \Delta_2^{\frac{m-2}{m}},\ t^{m-1} h_2 \Delta_2^{\frac{m-1}{m}}, \tag{55}$$
are severally equal, in some order, to those in (39). Therefore Δ_2 is equal to one of the terms
$$\Delta_1,\ a_1^m \Delta_1^2,\ \ldots,\ e_1^m \Delta_1^{m-2},\ h_1^m \Delta_1^{m-1}. \tag{56}$$
In like manner each of the terms in (54) is equal to a term in (56). And,

because the terms in (54) are unequal, they are severally equal to different terms in (56). By Prop. III, the terms in (54) are the roots of a rational irreducible equation, say $\psi_1(x)=0$. Rejecting from the series (56) the roots of the equation $\psi_1(x)=0$, certain of the remaining terms must in the same way be the roots of a rational irreducible equation $\psi_2(x)=0$. And so on. Ultimately, if $\phi(x)$ be the continued product of the expressions $\psi_1(x)$, $\psi_2(x)$, etc., the terms in (56) are the roots of the rational equation $\phi(x)=0$.

§53. The equations $\psi_1(x)=0$, $\psi_2(x)=0$, etc., formed by means of the expressions $\psi_1(x)$, $\psi_2(x)$, etc., may be said to be *sub-auxiliary* to the equation $F(x)=0$. It will be observed that the sub-auxiliaries are all irreducible.

§54. PROPOSITION XVIII. In passing from r_1 to r_n, while Δ_1 becomes Δ_n, the expressions a_1, b_1, etc., which, by Prop. XIV, involve only surds occurring in Δ_1, must severally receive determinate values, a_n, b_n, etc. In other words, a_1 being a particular cognate form of A, there cannot, for the same value of Δ_n, be two particular cognate forms of A, as a_n and a_N, unequal to one another. And so in the case of b_1, e_1, etc.

For, just as each of the terms in (42) is equal to a term in (41), there are primitive m^{th} roots of unity τ and T such that the expressions

$$\tau\Delta_n^{\frac{1}{m}}+\tau^2 a_n\Delta_n^{\frac{2}{m}}+\text{etc.},\quad T\Delta_N^{\frac{1}{m}}+T^2 a_N\Delta_N^{\frac{2}{m}}+\text{etc.},$$

are equal to one another. Therefore, if $\Delta_N=\Delta_n$, in which case, by assigning suitable values to τ and T, $\Delta_N^{\frac{1}{m}}$ may be taken to be equal to $\Delta_n^{\frac{1}{m}}$,

$$\Delta_n^{\frac{1}{m}}(\tau-T)+\Delta_n^{\frac{2}{m}}(a_n\tau^2-a_N T^2)+\text{etc.}=0. \tag{57}$$

Suppose if possible that the coefficients of the different powers of $\Delta_n^{\frac{1}{m}}$ in (57) are not all zero. Then, by §5, $t\Delta_n^{\frac{1}{m}}=l_1$; t being an m^{th} root of unity; and l_1 involving only surds of lower ranks than $\Delta_1^{\frac{1}{m}}$. Hence, by Prop. XV and Cor. Prop. XV, $\Delta_1^{\frac{1}{m}}$ is a rational function of surds of lower ranks than $\Delta_1^{\frac{1}{m}}$ and of the primitive m^{th} root of unity; which, by the definition in §6, is impossible. Since then the coefficients of the different powers of $\Delta_n^{\frac{1}{m}}$ in (57) are separately zero, $\tau=T$, $a_n\tau^2=a_N T^2$, therefore $a_n=a_N$.

§55. PROPOSITION XIX. Let the terms in (39) be written respectively

$$\Delta_1^{\frac{1}{m}},\ \delta_2^{\frac{1}{m}},\ \delta_3^{\frac{1}{m}},\ \ldots,\ \delta_{m-1}^{\frac{1}{m}}. \tag{58}$$

The symbols Δ_1, δ_2, δ_3, etc., are employed instead of Δ_1, Δ_2, Δ_3, because this latter notation might suggest, what is not necessarily true, that the terms in (56)

are all of them particular cognate forms of the generic expression under which Δ_1 falls. Then (compare Prop. XIII) the $m-1$ expressions in each of the groups

$$\left.\begin{array}{l} (\Delta_1^{\frac{1}{m}}\delta_{m-1}^{\frac{1}{m}},\ \delta_2^{\frac{1}{m}}\delta_{m-2}^{\frac{1}{m}},\ \delta_3^{\frac{1}{m}}\delta_{m-3}^{\frac{1}{m}},\ \ldots,\ \delta_{m-1}^{\frac{1}{m}}\Delta_1^{\frac{1}{m}},) \\ (\Delta_1^{\frac{2}{m}}\delta_{m-2}^{\frac{1}{m}},\ \delta_2^{\frac{2}{m}}\delta_{m-4}^{\frac{1}{m}},\ \delta_3^{\frac{2}{m}}\delta_{m-6}^{\frac{1}{m}},\ \ldots,\ \delta_{m-1}^{\frac{2}{m}}\delta_2^{\frac{1}{m}},) \\ (\Delta_1^{\frac{3}{m}}\delta_{m-3},\ \delta_2^{\frac{3}{m}}\delta_{m-6}^{\frac{1}{m}},\ \delta_3^{\frac{3}{m}}\delta_{m-9}^{\frac{1}{m}},\ \ldots,\ \delta_{m-1}^{\frac{3}{m}}\delta_3^{\frac{1}{m}},) \end{array}\right\} \tag{59}$$

and so on, are the roots of a rational equation of the $(m-1)^{\text{th}}$ degree. Also (compare Prop. X) the first $\frac{m-1}{2}$ terms in the first of the groups (59) are the roots of a rational equation of the $\left(\frac{m-1}{2}\right)^{\text{th}}$ degree.

In the enunciation of the proposition the remark is made that the series (54) is not necessarily identical with the series $\Delta_1, \delta_2, \delta_3, \ldots, \delta_{m-1}$. The former consists of the unequal particular cognate forms of Δ; the latter consists of the roots of the auxiliary equation $\phi(x)=0$. These two series are identical only when the auxiliary is irreducible. To prove the first part of the proposition, take the terms forming the second of the groups (59). Because $\delta_{m-2}^{\frac{1}{m}}$ represents $e_1\Delta_1^{\frac{m-2}{m}}$, $e_1\Delta_1=\Delta_1^{\frac{2}{m}}\delta_{m-2}^{\frac{1}{m}}$. Let E be the generic symbol under which the simplified expression e_1 falls. By Prop. XVIII, when Δ_1 is changed successively into the c terms in (54), e_1 receives successively the determinate values $e_1, e_2, \ldots, e_c$; and therefore $e_1\Delta_1$ receives successively the determinate values

$$e_1\Delta_1,\ e_2\Delta_2,\ \ldots,\ e_c\Delta_c. \tag{60}$$

There is therefore no particular cognate form of $E\Delta$ that is not equal to a term in (60). By Prop. XVI, there is a value of the m^{th} root of unity t for which the terms in (55) are severally equal, in some order, to those in (39). Let the term in (39) to which $t\Delta_2^{\frac{1}{m}}$ is equal be $q_1\Delta_1^{\frac{n}{m}}$. Then, applying the principle of Cor. Prop. XV, as in Prop. XVI, it follows that the term in (39) to which $t^{m-2}e_2\Delta_2^{\frac{m-2}{m}}$ in (55) is equal is $k_1\Delta_1^{\frac{M-2n}{m}}$, M being a multiple of m, and $M-2n$ being less than m. Therefore $e_2\Delta_2$ is equal to $q_1^2k_1\Delta_1^{\frac{M}{m}}$, which is the product of two of the terms in (39) occurring respectively at equal distances from opposite extremities of the series. In other words, $e_2\Delta_2$ is equal to an expression $\delta_n^{\frac{2}{m}}\delta_{m-2n}^{\frac{1}{m}}$ in the second of the groups (59). In like manner every term in (60) is equal to an expression in the second of the groups (59). Let the unequal terms in (60) be

$$e_1\Delta_1, \text{ etc.} \tag{61}$$

Then, by Prop. III, the terms in (61) are the roots of a rational irreducible

equation, say $f_1(x) = 0$. Rejecting these, which are distinct terms in the second of the groups (59), it can in like manner be shown that certain other terms in that group are the roots of a rational irreducible equation, say $f_2(x) = 0$. And so on. Ultimately, if $f(x)$ be the continued product of the expressions $f_1(x)$, $f_2(x)$, etc., the terms forming the second of the groups (59) are the roots of a rational equation of the $(m-1)^{\text{th}}$ degree. The proof applies substantially to each of the other groups. To prove the second part, it is only necessary to observe that, in the first of the groups (59), the last term is identical with the first, the last but one with the second, and so on.

§56. *Cor.* 1. The reasoning in the proposition proceeds on the assumption that the prime number m is odd. Should m be even, the series Δ_1, δ_1, etc., is reduced to its first term. The law may be considered even then to hold in the following form. The product $\Delta_1^{\frac{1}{m}} \Delta_1^{\frac{1}{m}}$ is the root of a rational equation of the $(m-1)^{\text{th}}$ degree, or is rational. For this product is Δ_1, which, by Prop. XVII, is the root of an equation of the $(m-1)^{\text{th}}$ degree.

§57. *Cor.* 2. I merely notice, without farther proof, that the generalization in §45 in the case when the equation $F(x) = 0$ is of the first (see §30) class holds in the present case likewise.

Analysis of Solvable Equations of the Fifth Degree.

§58. Let the solvable irreducible equation of the m^{th} degree, which we have been considering, be of the fifth degree. Then, by Prop. IX and §47, whether the equation belongs to the first or to the second of the two classes that have been distinguished, assuming the sum of the roots g to be zero,

$$r_1 = \tfrac{1}{5}(\Delta_1^{\frac{1}{5}} + a_1\Delta_1^{\frac{2}{5}} + e_1\Delta_1^{\frac{3}{5}} + h_1\Delta_1^{\frac{4}{5}}), \tag{62}$$

though, when the equation is of the first class, the root, as thus presented, is not in a simple state.

§59. Proposition XX. If the auxiliary biquadratic has a rational root Δ_1 not zero, all the roots of the auxiliary biquadratic are rational.

Because Δ_1 is rational, the auxiliary biquadratic $\phi(x) = 0$ is not irreducible. Therefore, by Prop. VII, the equation $F(x) = 0$ is of the second (see §30) class. Therefore, by Prop. XIV, $\Delta_1^{\frac{1}{5}}$ is the only principal surd in r_1. Consequently, because Δ_1 is rational, a_1, e_1 and h_1 are rational. Therefore Δ_1, $a_1^5\Delta_1^2$, $e_1^5\Delta_1^3$, $h_1^5\Delta_1^4$, which are the roots of the auxiliary biquadratic, are rational.

§60. PROPOSITION XXI. If the auxiliary biquadratic has a quadratic sub-auxiliary $\psi_1(x)=0$ with the roots Δ_1 and Δ_2, then $\Delta_2 = h_1^5\Delta_1^4$, and $\Delta_1 = h_2^5\Delta_2^4$; and $h_1\Delta_1$ is rational.

As in §52, t being a certain fifth root of unity, each term in (55) is equal to a term in (39). The first term in (55) cannot be equal to the first in (39), for this would make $\Delta_2 = \Delta_1$. Suppose if possible that the first in (55) is equal to the second in (39). Then, by equations (50), applied as in Prop. XVI,

$$\left.\begin{array}{ll} & t\Delta_2^{\frac{1}{5}} = a_1\Delta_1^{\frac{2}{5}},\ \ t^2a_2\Delta_2^{\frac{2}{5}} = h_1\Delta_1^{\frac{4}{5}},\ \ t^3e_2\Delta_2^{\frac{3}{5}} = \Delta_1^{\frac{1}{5}},\ \ t^4h_2\Delta_2^{\frac{4}{5}} = e_1\Delta_1^{\frac{3}{5}}, \\ \text{therefore} & \Delta_2 = a_1^5\Delta_1^2,\quad a_2^5\Delta_2^2 = h_1^5\Delta_1^4,\quad e_2^5\Delta_2^3 = \Delta_1,\quad h_2^5\Delta_2^4 = e_1^5\Delta_1^3. \end{array}\right\} \tag{63}$$

Now $a_1^5\Delta_1^2$, being equal to Δ_2, is a root of the equation $\psi_1(x)=0$. And $a_1^5\Delta_1^2$, involving only surds that occur in r_1, is in a simple state. Therefore, by Prop. III, $a_2^5\Delta_2^2$ is a root of the equation $\psi_1(x)=0$. Therefore $h_1^5\Delta_1^4$, and therefore also $h_2^5\Delta_2^4$ or $e_1^5\Delta_1^3$, are roots of that equation. Hence all the terms

$$\Delta_1,\ a_1^5\Delta_1^2,\ e_1^5\Delta_1^3,\ h_1^5\Delta_1^4, \tag{64}$$

are roots of the equation $\psi_1(x)=0$. But a_1, e_1, h_1, are all distinct from zero; for, by (63), if one of them was zero, all would be zero, and therefore $\Delta_1^{\frac{1}{5}}$ would be zero; which by §6, is impossible. From this it follows that no two terms in (64) are equal to one another; for, taking $a_1^5\Delta_1^2$ and $e_1^5\Delta_1^3$, if these were equal, we should have $e_1t'\Delta_1^{\frac{1}{5}} = a_1$, t' being a fifth root of unity; which, by §8, is impossible. This gives the equation $\psi_1(x)=0$ four unequal roots; which, because it is of the second degree, is impossible. Therefore the first term in (55) is not equal to the second in (39). In the same way it can be shown that it is not equal to the third. Therefore it must be equal to the fourth. In like manner the first in (39) is equal to the fourth in (55). Because then $t\Delta_2^{\frac{1}{5}} = h_1\Delta_1^{\frac{4}{5}}$, and $\Delta_1^{\frac{1}{5}} = t^4h_2\Delta_2^{\frac{4}{5}}$, $h_2\Delta_2 = h_1\Delta_1$. But, just as it was proved in §56 that, the roots of the sub-auxiliary $\psi_1(x)=0$ being the c terms Δ_1, Δ_2, etc., there is no particular cognate form of $E\Delta$ that is not a term in the series $e_1\Delta_1, e_2\Delta_2, \ldots, e_c\Delta_c$, it follows that, if h_1 be a particular cognate form of H, there is no particular cognate form of $H\Delta$ that is not equal to one of the terms $h_1\Delta_1$ and $h_2\Delta_2$. Hence, since $h_1\Delta_1 = h_2\Delta_2$, $H\Delta$ has no particular cognate form different in value from $h_1\Delta_1$. Therefore, by Prop. III, $h_1\Delta_1$ is rational.

§61. PROPOSITION XXII. The auxiliary biquadratic $\phi(x)=0$ either has all its roots rational, or has a sub-auxiliary (see §53) of the second degree, or is irreducible.

It will be kept in view that the sub-auxiliaries are, by the manner of their formation, irreducible. First, let the series (54), containing the roots of the sub-auxiliary $\psi_1(x)=0$ consist of a single term Δ_1. Then, by Prop. III, Δ_1 is rational. Therefore, by Prop. XX, all the roots of the auxiliary are rational. Next, let the series (54) consist of the two terms Δ_1 and Δ_2. By this very hypothesis, the auxiliary biquadratic has a quadratic sub-auxiliary. Lastly, let the series (54) contain more than two terms. Then it has the three terms Δ_1, Δ_2, Δ_3. We have shown that these must be severally equal to terms in (64). Neither Δ_2 nor Δ_3 is equal to Δ_1. They cannot both be equal to $h_1^5\Delta_1^4$. Therefore one of them is equal to one of the terms $a_1^5\Delta_1^2$, $e_1^5\Delta_1^3$. But in §60 it appeared that, if Δ_2 be equal either to $a_1^5\Delta_1^2$ or to $e_1^5\Delta_1^3$, all the terms in (64) are roots of the irreducible equation of which Δ_1 is a root. The same thing holds regarding Δ_3. Therefore, when the series (54) contains more than two terms, the irreducible equation which has Δ_1 for one of its roots has the four unequal terms in (64) for roots; that is to say, the auxiliary biquadratic is irreducible.

§62. Let $5u_1=\Delta_1^{\frac{1}{5}}$, $5u_2=a_1\Delta_1^{\frac{2}{5}}$, $5u_3=e_1\Delta_1^{\frac{3}{5}}$, $5u_4=h_1\Delta_1^{\frac{4}{5}}$; and, n being any whole number, let S_n denote the sum of the n^{th} powers of the roots of the equation $F(x)=0$. Then

$$S_1=0;\; S_2=10(u_1u_4+u_2u_3);\; S_3=15\{\Sigma(u_1u_2^2)\};$$
$$S_4=20\{\Sigma(u_1^3u_2)\}+30(u_1^2u_4^2+u_2^2u_3^2)+120\,u_1u_2u_3u_4;$$
$$S_5=5\{\Sigma(u_1^5)\}+100\{\Sigma(u_1^3u_3u_4)\}+150\{\Sigma(u_1u_3^2u_4^2)\};$$

where such an expression as $\Sigma(u_1u_2^2)$ means the sum of all such terms as $u_1u_2^2$; it being understood that, as any one term in the circle u_1, u_2, u_4, u_3, passes into the next, that next passes into its next, u_3 passing into u_1.

The Roots of the Auxiliary Biquadratic all Rational.

§63. Any rational values that may be assigned to Δ_1, a_1, e_1, and h_1 in r_1, taken as in (62), make r_1 the root of a rational equation of the fifth degree, for they render the values of S_1, S_2, etc., in §62, rational. In fact, $S_1=0$, $25S_2=10\Delta_1(h_1+a_1e_1)$, and so on.

The Auxiliary Biquadratic with a Quadratic Sub-Auxiliary.

§64. Proposition XXIII. In order that r_1, taken as in (62), may be the root of an irreducible equation $F(x)=0$ of the fifth degree, whose auxiliary biquadratic has a quadratic sub-auxiliary, it must be of the form

$$r_1=\tfrac{1}{5}\{(\Delta_1^{\frac{1}{5}}+\Delta_2^{\frac{1}{5}})+(a_1\Delta_1^{\frac{2}{5}}+a_2\Delta_2^{\frac{2}{5}})\}; \tag{65}$$

where Δ_1 and Δ_2 are the roots of the irreducible equation $\psi_1(x) = x^2 - 2px + q^5 = 0$; and $a_1 = b + d\surd(p^2 - q^5)$, $a_2 = b - d\surd(p^2 - q^5)$; p, q, b and d being rational; and the roots $\Delta_1^{\frac{1}{5}}$ and $\Delta_2^{\frac{1}{5}}$ being so related that $\Delta_1^{\frac{1}{5}}\Delta_2^{\frac{1}{5}} = q$.

By Prop. VII, when a quintic equation is of the first (see §30) class, the auxiliary biquadratic is irreducible. Hence, in the case which we are considering, the quintic is of the second class. The quadratic sub-auxiliary may be assumed to be $\psi_1(x) = x^2 - 2px + k = 0$, p and k being rational. By Prop. XXI, the roots of the equation $\psi_1(x) = 0$ are Δ_1 and $h_1^5\Delta_1^4$. Therefore $k = (h_1\Delta_1)^5$; or, putting q for $h_1\Delta_1$, $k = q^5$. By the same proposition, $h_1\Delta_1$ is rational. Therefore q is rational. Hence $\psi_1(x)$ has the form specified in the enunciation of the proposition. Next, by Proposition XVI, there is a fifth root of unity t such that $t\Delta_2^{\frac{1}{5}} = h_1\Delta_1^{\frac{4}{5}}$. If we take t to be unity, which we may do by a suitable interpretation of the symbol $\Delta_2^{\frac{1}{5}}$, $\Delta_2^{\frac{1}{5}} = h_1\Delta_1^{\frac{4}{5}}$ This implies that $e_1\Delta_1^{\frac{3}{5}} = a_2\Delta_2^{\frac{2}{5}}$, a_2 being what a_1 becomes in passing from Δ_1 to Δ_2. Substituting these values of $e_1\Delta_1^{\frac{3}{5}}$ and $h_1\Delta_1^{\frac{4}{5}}$ in (62), we obtain the form of r_1 in (65), while at the same time $\Delta_1^{\frac{1}{5}}\Delta_2^{\frac{1}{5}} = h_1\Delta_1 = q$. The forms of a_1 and a_2 have to be more accurately determined. By Prop. XIV, $\Delta_1^{\frac{1}{5}}$ is the only principal surd that r_1, as presented in (62), contains. Therefore a_1 involves no surd that does not occur in Δ_1; that is to say, $\surd(p^2 - q^5)$ is the only surd in a_1. Hence we may put $a_1 = b + d\surd(p^2 - q^5)$, b and d being rational. But a_2 is what a_1 becomes in passing from Δ_1 to Δ_2. And Δ_2 differs from Δ_1 only in the sign of the root $\surd(p^2 - q^5)$. Therefore $a_2 = b - d\surd(p^2 - q^5)$.

§65. Any rational values that may be assigned to b, d, p and q in r_1, taken as in (65), make r_1 the root of a rational equation of the fifth degree; for they render the values of S_1, S_2, etc., in §62, rational. In fact, $S_1 = 0$, $25S_2 = 10\{q + q^2b^2 - q^2d^2(p^2 - q^5)\}$, and so on.

The Auxiliary Biquadratic Irreducible.

§66. When the auxiliary biquadratic is irreducible, the unequal particular cognate forms of Δ are, by Prop. III, four in number, Δ_1, Δ_2, Δ_3, Δ_4. As explained in §55, because the equation $\phi(x) = 0$ is irreducible, these terms are severally identical with Δ_1, δ_2, δ_3, δ_4. Hence, putting $m = 5$, the first two terms in the first of the groups (59) may be written in the notation of (37),

$$\Delta_1^{\frac{1}{5}}\Delta_4^{\frac{1}{5}},\ \Delta_2^{\frac{1}{5}}\Delta_3^{\frac{1}{5}}; \tag{66}$$

and the second and third groups may be written

$$\left.\begin{array}{l}(\Delta_1^{\frac{2}{5}}\Delta_3^{\frac{1}{5}},\ \Delta_2^{\frac{2}{5}}\Delta_1^{\frac{1}{5}},\ \Delta_3^{\frac{2}{5}}\Delta_4^{\frac{1}{5}},\ \Delta_4^{\frac{2}{5}}\Delta_2^{\frac{1}{5}})\\ (\Delta_1^{\frac{3}{5}}\Delta_2^{\frac{1}{5}},\ \Delta_2^{\frac{3}{5}}\Delta_4^{\frac{1}{5}},\ \Delta_3^{\frac{3}{5}}\Delta_1^{\frac{1}{5}},\ \Delta_4^{\frac{3}{5}}\Delta_3^{\frac{1}{5}}).\end{array}\right\} \tag{67}$$

§67. Proposition XXIV. The roots of the auxiliary biquadratic equation $\phi(x)=0$ are of the forms

$$\left.\begin{array}{l}\Delta_1 = m+n\sqrt{z}+\sqrt{s},\ \Delta_2 = m-n\sqrt{z}+\sqrt{s_1},\\ \Delta_4 = m+n\sqrt{z}-\sqrt{s},\ \Delta_3 = m-n\sqrt{z}-\sqrt{s_1};\end{array}\right\} \tag{68}$$

where $s=p+q\sqrt{z}$, and $s_1=p-q\sqrt{z}$; m, n, z, p and q being rational; and the surd $\sqrt{s}$ being irreducible.

By Propositions XIII and XIX, the terms in (66) are the roots of a quadratic. Therefore $\Delta_1\Delta_4$ and $\Delta_2\Delta_3$ are the roots of a quadratic. Suppose if possible that $\Delta_1\Delta_3$ is the root of a quadratic. By Propositions IX and XIX, $\Delta_3^{\frac{1}{5}} = e_1\Delta_1^{\frac{3}{5}}$. Therefore $e_1^5\Delta_1^4$ is the root of a quadratic. From this it follows (Prop. III) that there are not more than two unequal terms in the series,

$$e_1^5\Delta_1^4,\ e_2^5\Delta_2^4,\ e_3^5\Delta_3^4,\ e_4^5\Delta_4^4. \tag{69}$$

But suppose if possible that $e_1^5\Delta_1^4 = e_2^5\Delta_2^4$. Then, t being one of the fifth roots of unity, $te_1\Delta_1^{\frac{4}{5}} = e_2\Delta_2^{\frac{4}{5}}$. But, by Propositions IX and XIX, $\Delta_2^{\frac{1}{5}} = h_1\Delta_1^{\frac{4}{5}}$. Therefore, $te_1\Delta_1^{\frac{4}{5}} = e_2h_1^4\Delta_1^3\Delta_1^{\frac{1}{5}}$. Therefore, by §8, $e_1=0$. Therefore one of the roots of the auxiliary biquadratic is zero; which because the auxiliary biquadratic is assumed to be irreducible, is impossible. Therefore $e_1^5\Delta_1^4$ and $e_2^5\Delta_2^4$ are unequal. In the same way all the terms in (69) can be shown to be unequal; which, because it has been proved that there are not more than two unequal terms in (69), is impossible. Therefore $\Delta_1\Delta_3$ is not the root of a quadratic equation. Therefore the product of two of the roots, Δ_1 and Δ_4, of the auxiliary biquadratic is the root of a quadratic equation, while the product of a different pair, Δ_1 and Δ_3, is not the root of a quadratic. But the only forms which the roots of an irreducible biquadratic can assume consistently with these conditions are those given in (68).

§68. Proposition XXV. The surd $\sqrt{s_1}$ can have its value expressed in terms of $\sqrt{s}$ and $\sqrt{z}$.

By Propositions XIII and XIX, the terms of the first of the groups (67) are the roots of a biquadratic equation. Therefore their fifth powers

$$\Delta_1^2\Delta_3,\ \Delta_2^2\Delta_1,\ \Delta_3^2\Delta_4,\ \Delta_4^2\Delta_2, \tag{70}$$

are the roots of a biquadratic. From the values of Δ_1, Δ_2, Δ_3 and Δ_4 in (68),

the values of the terms in (70) may be expressed as follows:

$$\left.\begin{aligned}
\Delta_1^2\Delta_3 &= F+F_1\sqrt{z}+(F_2+F_3\sqrt{z})\sqrt{s}+(F_4+F_5\sqrt{z})\sqrt{s_1}+(F_6+F_7\sqrt{z})\sqrt{s}\sqrt{s_1},\\
\Delta_2^2\Delta_1 &= F-F_1\sqrt{z}+(F_2-F_3\sqrt{z})\sqrt{s_1}-(F_4-F_5\sqrt{z})\sqrt{s}-(F_6-F_7\sqrt{z})\sqrt{s}\sqrt{s_1},\\
\Delta_4^2\Delta_2 &= F-F_1\sqrt{z}-(F_2-F_3\sqrt{z})\sqrt{s_1}+(F_4-F_5\sqrt{z})\sqrt{s}-(F_6-F_7\sqrt{z})\sqrt{s}\sqrt{s_1},\\
\Delta_3^2\Delta_4 &= F+F_1\sqrt{z}-(F_2+F_3\sqrt{z})\sqrt{s}-(F_4+F_5\sqrt{z})\sqrt{s_1}+(F_6+F_7\sqrt{z})\sqrt{s}\sqrt{s_1},
\end{aligned}\right\} \quad (71)$$

where F, F_1, etc., are rational. Let $\Sigma(\Delta_1^2\Delta_3)$ be the sum of the four expressions in (70). Then, because these expressions are the roots of a biquadratic, $\Sigma(\Delta_1^2\Delta_3)$ or $4F+4F_7\sqrt{s}\sqrt{s_1}$ must be rational. Suppose if possible that $\sqrt{s_1}$ cannot have its value expressed in terms of $\sqrt{s}$ and $\sqrt{z}$. Then, because $\sqrt{s}\sqrt{s_1}$ is not rational, $F_7=0$. By (68), this implies that $n=0$. Let

$(\Delta_1^2\Delta_3)^2=L+L_1\sqrt{z}+(L_2+L_3\sqrt{z})\sqrt{s}+(L_4+L_5\sqrt{z})\sqrt{s_1}+(L_6+L_7\sqrt{z})\sqrt{s}\sqrt{s_1}$,

where L, L_1, etc., are rational. Then, as above, $L_7=0$. Keeping in view that $n=0$, this means that $m^2q=0$. But q is not zero, for this would make $\sqrt{s}=\sqrt{s_1}$; which, because we are reasoning on the hypothesis that $\sqrt{s_1}$ cannot have its value expressed in terms of $\sqrt{s}$ and $\sqrt{z}$, is impossible. Therefore m is zero. And it was shown that n is zero. Therefore $\Delta_1=\sqrt{s}$, and $\Delta_3=-\sqrt{s}$. Therefore $\Delta_1\Delta_3=-\sqrt{(p^2-q^2z)}$; which, because it has been proved that $\Delta_1\Delta_3$ is not the root of a quadratic equation, is impossible. Hence $\sqrt{s_1}$ cannot but be a rational function of $\sqrt{s}$ and $\sqrt{z}$.

§69. PROPOSITION XXVI. The form of s is

$$h(1+e^2)+h\sqrt{(1+e^2)}, \qquad (72)$$

h and e being rational, and $1+e^2$ being the value of z.

By Prop. XXV, $\sqrt{s_1}=v+c\sqrt{s}$, v and c being rational functions of $\sqrt{z}$. Therefore $s_1=v^2+c^2s+2vc\sqrt{s}$. By Prop. XXIV, $\sqrt{s}$ is irreducible. Therefore $vc=0$. But c is not zero, for this would make $\sqrt{s_1}=v$, and thus $\sqrt{s_1}$ would be the root of a quadratic equation. Therefore $v=0$, and $\sqrt{s_1}=c\sqrt{s}$ $=(c_1+c_2\sqrt{z})\sqrt{s}$, c_1 and c_2 being rational. Therefore

$$\begin{aligned}\sqrt{(ss_1)}=\sqrt{(p^2-q^2z)}&=(c_1+c_2\sqrt{z})(p+q\sqrt{z})\\ &=(c_1p+c_2qz)+\sqrt{z}(c_1q+c_2p)=P+Q\sqrt{z}.\end{aligned}$$

Here, since p^2-q^2z is rational, either $P=0$ or $Q=0$. As the latter of these alternatives would make $\sqrt{(p^2-q^2z)}$ rational, and therefore would make $\sqrt{(p+q\sqrt{z})}$ or $\sqrt{s}$ reducible, it is inadmissible. Therefore $c_1p+c_2qz=0$, and $\sqrt{(p^2-q^2z)}=(c_1q+c_2p)\sqrt{z}$. Now qz is not zero, for this would make $\sqrt{(ss_1)}=\pm p$; which, because $\sqrt{s}$ is irreducible, is impossible. Therefore $c_2=0$. But, by hypothesis, $c_1=0$; therefore $\sqrt{s_1}$, which is equal to $(c_1+c_2\sqrt{z})\sqrt{s}$, is

zero; which is impossible. Hence c_1 cannot be zero. We may therefore put $ce = 1$, and $h(1 + e^2) = p$. Then $s = p + q\surd z = h(1 + e^2) + h\surd(1 + e^2)$. Having obtained this form, we may consider z to be identical with $1 + e^2$, q with h, and p with $h(1 + e^2)$.

§70. The reasoning in the preceding section holds good whether the equation $F(x) = 0$ be of the first (see §30) or of the second class. If we had had to deal simply with equations of the first class, the proof given would have been unnecessary, so far as the form of z is concerned; because, in that case, by Prop. VIII, Δ_1 is a rational function of the primitive fifth root of unity.

§71. Proposition XXVII. Under the conditions that have been established, the root r_1 takes the form given without deduction in *Crelle* (Vol. V, p. 336) from the papers of Abel.

For, by *Cor.* Prop. XIV (compare also *Cor.* 2, Prop. XIX), the expressions

$$\Delta_1^{\frac{1}{5}}\Delta_3^{\frac{2}{5}}\Delta_4^{\frac{3}{5}}\Delta_2^{\frac{3}{5}},\quad \Delta_2^{\frac{1}{5}}\Delta_1^{\frac{2}{5}}\Delta_3^{\frac{3}{5}}\Delta_4^{\frac{3}{5}},\quad \Delta_3^{\frac{1}{5}}\Delta_4^{\frac{2}{5}}\Delta_2^{\frac{3}{5}}\Delta_1^{\frac{3}{5}},\quad \Delta_4^{\frac{1}{5}}\Delta_2^{\frac{2}{5}}\Delta_1^{\frac{3}{5}}\Delta_3^{\frac{3}{5}}, \tag{73}$$

are the roots of a biquadratic equation. In the corollaries referred to, it is merely stated that each of the expressions in (73) is the root of a biquadratic; but the principles of the propositions to which the corollaries are attached show that the four expressions must be the roots of the same biquadratic. Let the terms in (73) be denoted respectively by

$$5A_1^{-1},\qquad 5A_2^{-1},\qquad 5A_3^{-1},\qquad 5A_4^{-1}.$$

Then $\Delta_1^{\frac{1}{5}}\Delta_3^{\frac{2}{5}}\Delta_4^{\frac{4}{5}}\Delta_2^{\frac{3}{5}} = \Delta_4^{\frac{1}{5}}(\Delta_1^{\frac{1}{5}}\Delta_3^{\frac{2}{5}}\Delta_4^{\frac{3}{5}}\Delta_2^{\frac{3}{5}})$ is an identity. Therefore

$$\tfrac{1}{5}\Delta_4^{\frac{1}{5}} = A_1(\Delta_1^{\frac{1}{5}}\Delta_3^{\frac{2}{5}}\Delta_4^{\frac{4}{5}}\Delta_2^{\frac{3}{5}}).\quad \text{Similarly, } \tfrac{1}{5}\Delta_3^{\frac{1}{5}} = A_3(\Delta_3^{\frac{1}{5}}\Delta_4^{\frac{2}{5}}\Delta_2^{\frac{4}{5}}\Delta_1^{\frac{3}{5}})$$

$$\tfrac{1}{5}\Delta_2^{\frac{1}{5}} = A_2(\Delta_2^{\frac{1}{5}}\Delta_1^{\frac{2}{5}}\Delta_3^{\frac{4}{5}}\Delta_4^{\frac{3}{5}}),\qquad \text{and } \tfrac{1}{5}\Delta_1^{\frac{1}{5}} = A_4(\Delta_4^{\frac{1}{5}}\Delta_2^{\frac{2}{5}}\Delta_1^{\frac{4}{5}}\Delta_3^{\frac{3}{5}}).$$

Substituting these values in (62), we get

$$r_1 = A_1(\Delta_1^{\frac{1}{5}}\Delta_3^{\frac{2}{5}}\Delta_4^{\frac{4}{5}}\Delta_2^{\frac{3}{5}}) + A_2(\Delta_2^{\frac{1}{5}}\Delta_1^{\frac{2}{5}}\Delta_3^{\frac{4}{5}}\Delta_4^{\frac{3}{5}}) + A_3(\Delta_3^{\frac{1}{5}}\Delta_4^{\frac{2}{5}}\Delta_2^{\frac{4}{5}}\Delta_1^{\frac{3}{5}}) + A_4(\Delta_4^{\frac{1}{5}}\Delta_2^{\frac{2}{5}}\Delta_1^{\frac{4}{5}}\Delta_3^{\frac{3}{5}}). \tag{74}$$

This, with immaterial differences in the subscripts, is Abel's expression; only we need to determine A_1, A_2, A_3 and A_4 more exactly. These terms are the reciprocals of the terms in (73) severally divided by 5. Therefore they are the roots of a biquadratic. Also, no surds can appear in A_1 except those that are present in Δ_1, Δ_2, Δ_3 and Δ_4. That is to say, A_1 is a rational function of $\surd s$, $\surd s_1$ and $\surd z$. But it was shown that $\surd s_1 \surd s = he\surd z$. Therefore A_1 is a rational function of $\surd s$ and $\surd z$. We may therefore put

$$A_1 = K + K'\Delta_1 + K''\Delta_4 + K'''\Delta_1\Delta_4.$$

K, K', K'' and K''' being rational. But the terms A_1, A_2, A_4, A_3 circulate with

Δ_1, Δ_2, Δ_4, Δ_3. Therefore

$$A_2 = K + K'\Delta_2 + K''\Delta_3 + K'''\Delta_2\Delta_3,$$
$$A_4 = K + K'\Delta_4 + K''\Delta_1 + K'''\Delta_1\Delta_4,$$
$$A_3 = K + K'\Delta_3 + K''\Delta_2 + K'''\Delta_2\Delta_3.$$

These are Abel's values.

§72. Keeping in view the values of Δ_1, Δ_2, etc., in (67), and also that $z = 1 + e^2$, and $s = hz + h\sqrt{z}$, any rational values that may be assigned to m, n, e, h, K, K', K'' and K''' make r_1, as presented in (74), the root of an equation of the fifth degree. For, any rational values of m, n, etc., make the values of S_1, S_2, etc., in §62, rational.

§73. It may be noted that, not only is the expression for r_1 in (74) the root of a quintic equation whose auxiliary biquadratic is irreducible, but, on the understanding that the surds $\sqrt{s}$ and $\sqrt{z}$ in Δ_1 may be reducible, the expression for r_1 in (74) contains the roots both of all equations of the fifth degree whose auxiliary biquadratics have their roots rational, and of all that have quadratic sub-auxiliaries. It is unnecessary to offer proof of this.

§74. The equation $x^5 - 10x^3 + 5x^2 + 10x + 1 = 0$ is an example of a solvable quintic with its auxiliary biquadratic irreducible. One of its roots is

$$\omega^{\frac{1}{5}} + \omega\omega^{\frac{2}{5}} + \omega^3\omega^{\frac{3}{5}} + \omega^4\omega^{\frac{4}{5}},$$

ω being a primitive fifth root of unity. It is obvious that this root satisfies all the conditions that have been pointed out in the preceding analysis as necessary. A root of an equation of the seventh degree of the same character is

$$\omega^{\frac{1}{7}} + \omega^4\omega^{\frac{2}{7}} + \omega^4\omega^{\frac{3}{7}} + \omega^2\omega^{\frac{4}{7}} + \omega^2\omega^{\frac{5}{7}} + \omega^6\omega^{\frac{6}{7}},$$

ω being a primitive seventh root of unity. The general form under which these instances fall can readily be found. Take the cycle that contains all the primitive $(m^2)^{\text{th}}$ roots of unity,

$$\theta,\ \theta^{\beta},\ \theta^{\beta^2},\ \text{etc.,} \qquad (75)$$

m being prime. The number of terms in the cycle is $(m-1)^2$. Let θ_1 be the $(m+1)^{\text{th}}$ term in the cycle (75), θ_2 the $(2m+1)^{\text{th}}$ term, and so on. Then the root of an equation of the m^{th} degree, including the instances above given, is

$$r_1 = (\theta + \theta^{-1}) + (\theta_1 + \theta_1^{-1}) + \ldots + (\theta_{\frac{m-3}{2}} + \theta_{\frac{m-3}{2}}^{-1}).$$

Resolution of Solvable Equations of the Fifth Degree.

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§1. By means of the laws established in the paper entitled "Principles of the Solution of Equations of the Higher Degrees," which is concluded in the present issue of the Journal of Mathematics, a criterion of the solvability of equations of the fifth degree may be found, and the roots of solvable quintics obtained in terms of given numerical coefficients. In certain classes of cases, the roots can be determined in terms of coefficients to which particular numerical values have not been assigned, but which are only assumed to be so related as to make the equations solvable.

SKETCH OF THE METHOD EMPLOYED.

§2. Let r_1, r_2, r_3, r_4, r_5, be the roots of the solvable irreducible equation of the fifth degree wanting the second term,

$$F(x) = x^5 + p_2 x^3 + p_3 x^2 + p_4 x + p_5 = 0. \tag{1}$$

It was proved in the "Principles" that

$$r_1 = \tfrac{1}{5}(\Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}} + \Delta_3^{\frac{1}{5}} + \Delta_4^{\frac{1}{5}}),$$

where Δ_1, Δ_2, Δ_3, Δ_4 are the roots of a biquadratic equation auxiliary to the equation $F(x) = 0$. It was also shown that the root can be expressed in the form

$$r_1 = \tfrac{1}{5}(\Delta_1^{\frac{1}{5}} + a_1 \Delta_1^{\frac{2}{5}} + e_1 \Delta_1^{\frac{3}{5}} + h_1 \Delta_1^{\frac{4}{5}}), \tag{2}$$

where a_1, e_1, h_1, involve only surds occurring in Δ_1; and no surds occur in Δ_1 except $\sqrt{(hz + h\sqrt{z})}$ and its subordinate $\sqrt{z}$; z being equal to $1 + e^2$, and h and e being rational. As in the "Principles," we may put $5u_1 = \Delta_1^{\frac{1}{5}}$, $5u_2 = \Delta_2^{\frac{1}{5}}$, $5u_3 = \Delta_3^{\frac{1}{5}}$, $5u_4 = \Delta_4^{\frac{1}{5}}$. Then

$$r_1 = u_1 + u_2 + u_3 + u_4. \tag{3}$$

Let S_1 be the sum of the roots of the equation $F(x) = 0$, S_2 the sum of their squares, and so on. Also let

$$\left.\begin{aligned}
\Sigma(u_1^2 u_3) &= u_1^2 u_3 + u_2^2 u_1 + u_3^2 u_4 + u_4^2 u_2,\\
\Sigma(u_1^3 u_2) &= u_1^3 u_2 + u_2^3 u_4 + u_3^3 u_1 + u_4^3 u_3.\\
\Sigma(u_1 u_3^2 u_4^2) &= u_1 u_3^2 u_4^2 + u_2 u_1^2 u_3^2 + u_3 u_4^2 u_2^2 + u_4 u_2^2 u_1^2,\\
\Sigma(u_1^5) &= u_1^5 + u_2^5 + u_3^5 + u_4^5.\\
\text{Then}\qquad S_2 &= 10(u_1 u_4 + u_2 u_3),\ S_3 = 15\{\Sigma(u_1^2 u_3)\},\\
S_4 &= 20\{\Sigma(u_1^3 u_2)\} + \tfrac{3}{10}(S_2^2) + 60 u_1 u_2 u_3 u_4,\\
S_5 &= 5\{\Sigma(u_1^5)\} + \tfrac{2}{3}(S_2 S_3) + 50\{\Sigma(u_1 u_3^2 u_4^2)\}.
\end{aligned}\right\} \tag{4}$$

§3. It was proved in the "Principles" that $u_1 u_4$ and $u_2 u_3$ are the roots of a quadratic equation. But $25 u_1 u_4 = h_1 \Delta_1$, and $25 u_2 u_3 = a_1 e_1 \Delta_1$. Therefore, because a_1, e_1, h_1, involve no surds that are not subordinate to $\Delta_1^{\frac{1}{5}}$, $\sqrt{z}$ is the only surd that can appear in $u_1 u_4$ and $u_2 u_3$. Consequently we may put

$$u_1 u_4 = g + a\sqrt{z}, \text{ and } u_2 u_3 = g - a\sqrt{z}, \tag{5}$$

where g, a, are rational. It scarcely needs to be pointed out that these forms are valid whether the surd $\sqrt{z}$ is irreducible or not. Now $S_2 = 10(u_1 u_4 + u_2 u_3) = -2p_2$. Therefore

$$g = -\tfrac{1}{10}(p_2). \tag{6}$$

Again, it was shown in the "Principles" that the four expressions $u_1^2 u_3$, $u_2^2 u_1$, $u_3^2 u_4$, $u_4^2 u_2$, are the roots of a biquadratic equation. And, by the same reasoning as that employed in the case of $u_1 u_4$ and $u_2 u_3$, the only surds that can appear in

these expressions are $\sqrt{(hz+h\sqrt{z})}$, $\sqrt{(hz-h\sqrt{z})}$, and $\sqrt{z}$. Let $hz+h\sqrt{z}=s$, and $hz-h\sqrt{z}=s_1$. Then

$$\sqrt{s_1}=\left(\frac{\sqrt{z}-1}{e}\right)\sqrt{s}, \text{ and } \sqrt{s}\sqrt{s_1}=he\sqrt{z}. \tag{7}$$

Hence the expressions $u_1^2u_3$, $u_2^2u_1$, $u_3^2u_4$, $u_4^2u_2$, may have their value exhibited in terms of $\sqrt{z}$ and either of the surds $\sqrt{s}$, $\sqrt{s_1}$. Put

$$\left.\begin{aligned} u_1^2u_3 &= k+c\sqrt{z}+(\theta+\phi\sqrt{z})\sqrt{s},\\ u_4^2u_2 &= k+c\sqrt{z}-(\theta+\phi\sqrt{z})\sqrt{s},\\ u_2^2u_1 &= k-c\sqrt{z}+(\theta-\phi\sqrt{z})\sqrt{s_1},\\ u_3^2u_4 &= k-c\sqrt{z}-(\theta-\phi\sqrt{z})\sqrt{s_1}; \end{aligned}\right\} \tag{8}$$

where k, c, θ, ϕ, are rational. These coefficients must bear a relation to g, a, in (5). In fact, because $(u_1^2u_3)(u_4^2u_2)=(u_1u_4)^2(u_2u_3)$,

$$(g^2-a^2z)(g+a\sqrt{z})=(k+c\sqrt{z})^2-(\theta+\phi\sqrt{z})^2(hz+h\sqrt{z}).$$

Equating the rational parts to one another, and also the irrational parts,

$$\left.\begin{aligned} hz(\theta^2+\phi^2z+2\theta\phi) &= k^2+c^2z-g(g^2-a^2z),\\ h(\theta^2+\phi^2z+2\theta\phi z) &= 2kc-a(g^2-a^2z). \end{aligned}\right\} \tag{9}$$

Because $s_2=15\{\Sigma(u_1^2u_3)\}=-3p_3$,

$$k=-\tfrac{1}{20}(p_3). \tag{10}$$

It will be convenient to retain the symbols g and k, whose values are given in (6) and (10). Again, because $u_1^3u_2=\frac{(u_1^2u_3)(u_2^2u_1)}{u_2u_3}$, we have, from (5) and (8),

$$\begin{aligned} u_1^3u_2 &= \frac{g+a\sqrt{z}}{g^2-a^2z}\{k+c\sqrt{z}+(\theta+\phi\sqrt{z})\sqrt{s}\}\{k-c\sqrt{z}+(\theta-\phi\sqrt{z})\sqrt{s_1}\}\\ &= A+A'\sqrt{z}+(A''+A'''\sqrt{z})\sqrt{s}, \end{aligned}$$

where A, A', A'', A''', are rational. The value of A is

$$A=\frac{1}{g^2-a^2z}\{g(k^2-c^2z)+ahez(\theta^2-\phi^2z)\}. \tag{11}$$

Again, $u_1^5=\frac{(u_1^2u_3)^2(u_2^2u_1)}{(u_2u_3)^2}$. That is, from (8) and (5) and (7),

$$\begin{aligned} u_1^5=\frac{(g+a\sqrt{z})^2}{(g^2-a^2z)^2}&\{2(k+c\sqrt{z})^2-(g^2-a^2z)(g+a\sqrt{z})+2(k+c\sqrt{z})(\theta+\phi\sqrt{z})\sqrt{s}\}\\ &\{k-c\sqrt{z}+(\theta-\phi\sqrt{z})\sqrt{s_1}\}=B+B'\sqrt{z}+(B''+B'''\sqrt{z})\sqrt{s}; \end{aligned}$$

where B, B', B'', B''', are rational. Now, by (4),

$$S_4=20\{\Sigma(u_1^3u_2)\}+\tfrac{3}{10}(S_2^2)+60u_1u_2u_3u_4.$$

And $S_4=2p_2^2-4p_4$. Also $\Sigma(u_1^3u_2)=4A$; and, by (6), $10g=-p_2$; and, by (5), $u_1u_2u_3u_4=g^2-a^2z$. Therefore

$$p_4=-20A+5g^2+15a^2z. \tag{12}$$

Again, $S_5 = 5\{\Sigma(u_1^5)\} + \frac{2}{3}(S_2 S_3) + 50\{\Sigma(u_1 u_3^2 u_4^2)\}$.

And $\Sigma(u_1^5) = 4B$, $S_2 S_3 = 6p_2 p_3 = 1200gk$, and

$$\Sigma(u_1 u_3^2 u_4^2) = u_1 u_4(u_2^2 u_1 + u_3^2 u_4) + u_2 u_3(u_1^2 u_3 + u_4^2 u_2).$$

Therefore $S_5 = 20B + 1000gk - 200acz$.

But $S_5 - 5p_2 p_3 + 5p_5 = S_5 - 1000gk + 5p_5 = 0$.

Therefore
$$p_5 = -4B + 40acz. \tag{13}$$

The values of p_4 and p_5 in (12) and (13) make the quintic

$$F(x) = x^5 + p_2 x^3 + p_3 x^2 + (5g^2 + 15a^2 z - 20A)x + (40acz - 4B) = 0. \tag{14}$$

§4. Assuming the coefficients p_2, p_3, etc., in (1), to be known, the coefficients in the equation $F(x) = 0$ as exhibited in (14) involve six unknown quantities, namely, a, c, θ, ϕ, e, h. The list does not include z, g, k; because $z = 1 + e^2$; and g and k are known by (6) and (10). To find the six unknown quantities we have six equations, which are here gathered together.

$$\left.\begin{aligned} p_4 &= -20A + 5g^2 + 15a^2 z, \\ p_5 &= -4B + 40acz, \\ B'' &= 1, \\ B''' &= 0, \\ hz(\theta^2 + \phi^2 z + 2\theta\phi) &= k^2 + c^2 z - g(g^2 - a^2 z) \\ h(\theta^2 + \phi^2 z + 2\theta\phi z) &= 2kc - a(g^2 - a^2 z). \end{aligned}\right\} \tag{15}$$

The first two of these equations are the equations (12) and (13). As to the third and fourth, it was proved in the "Principles" that the form of u_1^5 is $m + n\sqrt{z} + \sqrt{(hz + h\sqrt{z})}$, m and n being rational. This is saying in other words that $B'' = 1$ and $B''' = 0$. The last two of the equations (15) are the equations (9).

§5. The criterion of solvability of the equation $F(x) = 0$ may now be stated in a general way to be that the coefficients p_2, p_3, etc., must be so related that rational quantities, a, c, θ, ϕ, e, h, exist satisfying the equations (15). We also see what requires to be done in order to find the roots of the equation $F(x) = 0$ in terms of the given coefficients. By (3), r_1 is known when u_1, u_2, u_3, u_4 are known. But, B'' and B''' being respectively unity and zero,

$$u_1^5 = B + B'\sqrt{z} + \sqrt{s}, \quad u_2^5 = B - B'\sqrt{z} + \sqrt{s_1},$$
$$u_4^5 = B + B'\sqrt{z} - \sqrt{s}, \quad u_3^5 = B - B'\sqrt{z} - \sqrt{s_1}.$$

Therefore, to find r_1 we need to find B, B', z and h; which is equivalent to saying that we need to find the six unknown quantities a, c, θ, ϕ, e, h. Before pointing out how this may be done in the most general case, I will refer to some special forms of soluble quintics.

CASE IN WHICH $u_1u_4 = u_2u_3$.

§6. A notable class of solvable quintics is that in which $u_1u_4 = u_2u_3$. It includes, as was proved in the "Principles," all the Gaussian equations of the fifth degree for the reduction of $x^n - 1 = 0$, n prime. It includes also other equations, of which examples will presently be given. Now, when $u_1u_4 = u_2u_3$, the root of the quintic can be found in terms of the coefficients p_2, p_3, etc., even while these coefficients retain their general symbolic forms; in other words, the root can be found in terms of p_2, p_3, etc., without definite numerical values being assigned to p_2, p_3, etc. This I proceed to show.

§7. By (5), because $u_1u_4 = u_2u_3$, $a = 0$. Thus, one of the six unknown quantities is determined, while we have still the six equations (15) to work with. It might be sufficient to say, that, from six equations five unknown rational quantities can be found. I will recur to this idea; but in the meantime the following line of reasoning may be pursued. From (11), $A = \frac{k^2 - c^2 z}{g}$. Therefore equation (12) becomes

$$gp_4 = -20(k^2 - c^2 z) + 5g^3. \tag{16}$$

Also, because $a = 0$, equations (7) being kept in view,

$$u_1^5 = \frac{1}{g^2}\{2(k^2 - c^2 z)(k + c\sqrt{z}) - g^3(k - c\sqrt{z}) + 2(k + c\sqrt{z})(\theta^2 - \phi^2 z)he\sqrt{z}\}$$
$$+ (B'' + B'''\sqrt{z})\sqrt{s}.$$

$$\therefore\ Bg^2 = k\{2(k^2 - c^2 z) - g^3\} + 2chez(\theta^2 - \phi^2 z)$$

and $$B'g^2 = c\{2(k^2 - c^2 z) + g^3\} + 2khe(\theta^2 - \phi^2 z);$$

$$\therefore\ u_1^5 = \frac{1}{g^2}[k\{2(k^2 - c^2 z) - g^3\} + 2chez(\theta^2 - \phi^2 z)]$$
$$+ \frac{\sqrt{z}}{g^2}[c\{2(k^2 - c^2 z) + g^3\} + 2khe(\theta^2 - \phi^2 z)] + \sqrt{s}. \tag{17}$$

Substitute in the second of equations (15) the value of B that has been obtained. Then

$$g^2 p_5 = -4k\{2(k^2 - c^2 z) - g^3\} - 8chez(\theta^2 - \phi^2 z). \tag{18}$$

The values of B'' and B''' are

$$\left.\begin{aligned} B''eg^2 &= \theta\{M + 2e(k^2 - c^2 z)\} - \phi zN = eg^2, \\ B'''eg^2 &= \theta N - \phi\{M - 2e(k^2 - c^2 z)\} = 0; \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} &\text{where } M = -2(k^2 + c^2 z) + g^3 + 4kcz, \text{ which may be written } M = 4kcz - P, \\ &\text{and } N = 2(k^2 + c^2 z) - g^3 - 4kc, \text{ which may be written } N = P - 4kc. \end{aligned}\right\} \tag{20}$$

The two equations (19) give us

$$\left.\begin{aligned} &\theta\{M^2 - zN^2 - 4e^2(k^2 - c^2 z)^2\} = eg^2\{M - 2e(k^2 - c^2 z)\}, \\ &\phi\{M^2 - zN^2 - 4e^2(k^2 - c^2 z)^2\} = eg^2 N. \\ &\text{Therefore}\quad \frac{\theta}{\varphi} = \frac{M - 2e(k^2 - c^2 z)}{N}. \end{aligned}\right\} \tag{21}$$

Equating the value of $\dfrac{\theta^2+\varphi^2 z+2\theta\varphi}{\theta^2+\varphi^2 z+2\theta\varphi z}$ obtained from (21) with that derived from the last two of equations (15),

$$\frac{k^2+c^2z-g^3}{2kcz}=\frac{\{M-2e(k^2-c^2z)\}^2+N^2z+2N\,\{M-2e(k^2-c^2z)\}}{\{M-2e(k^2-c^2z)\}^2+N^2z+2Nz\,\{M-2e(k^2-c^2z)\}}\,. \tag{22}$$

The coefficients p_2, p_3, etc., in the equation $F(x)=0$, being given, g and k are known by (6) and (10). Therefore, by (16), c^2z is known. Then (22) will be found to be a quadratic equation determinative of c. For, keeping in view the value of P in (20), (22) may be written

$$\frac{k^2+c^2z-g^3}{2kc^2z}=\frac{\{4(k^2+c^2z)^2+P^2\}-8kPc-16k(k^2-c^2z)(ce)}{\{4(k^2-c^2z)^2-16k^2c^2z-P^2\}\,c+8kc^2zP-4(k^2-c^2z)P(ce)}\,.$$

Because g, k, c^2z and P are known, this equation is of the form

$$H(ce)=Kc+L,$$

where H, K, L, are known. Therefore, since $c^2e^2=c^2z-c^2$,

$$c^2(H^2+K^2)+2KLc+(L^2-H^2c^2z)=0;$$

from which c is known. Therefore, since c^2z is known, z is known. Therefore e is known. Therefore, by (21), θ and ϕ are known. Therefore, by (18) or either of the equations (9), h is known. Therefore, by (17), u_1^5 is known. In like manner, u_2^5, u_3^5, u_4^5, are known. Hence finally, by (3), r_1 is known.

§8. *Example First.* I will now give some numerical verifications of the theory. The Gaussian equation of the fifth degree for the reduction of $x^{11}-1=0$, when deprived of its second term, is

$$x^5-\frac{22}{5}x^3-\frac{11}{25}x^2+\frac{11\times 42}{125}x+\frac{11\times 89}{3125}=0.$$

When a root of this equation is expressed as in (1), the value of r_1, as given by Lagrange, is

$$u_1^5=\frac{11}{4(5)^5}\{-89-25\sqrt{5}+5(19-9\sqrt{5})(-5-2\sqrt{5})\};$$

which, reduced to the form that we have adopted, is

$$u_1^5=\frac{11}{4(5)^5}\left\{-89+25\times\frac{89}{41}\sqrt{\left(\frac{5\times 41^2}{89^2}\right)}\right\}+\sqrt{(hz+h\sqrt{z})};$$

where $h=-\dfrac{11^2\times 89^2}{8\times 41\times(5)^8}$, $\sqrt{z}=-\dfrac{41}{89}\sqrt{5}$, and $e=-\dfrac{22}{89}$. We have to show that this is the result to which the equations of the preceding section lead. The simplest way will be to find g, k and c^2z by means of (6), (10) and (16), and then to take the values of e and $\sqrt{z}$ given above, and to substitute them in equation (22). If the theory is sound, the equation ought in this way to be

satisfied. When this equation has been satisfied, it will be unnecessary to pursue the verification farther. Because $p_2 = -\frac{22}{5}$, and $p_3 = -\frac{11}{25}$, $g = \frac{11}{25}$ and $k = \frac{11}{20 \times 25}$. From (18), taken in connection with (21), che must be negative. Therefore

$$c = -\frac{11 \times 89}{4 \times 25 \times 41},\ kc = -\frac{89}{80 \times 41}\left(\frac{11}{25}\right)^2,\ kcz = -\frac{41}{16 \times 89}\left(\frac{11}{25}\right)^2,$$

$$k^2 - c^2 z = -\frac{31}{100}\left(\frac{11}{25}\right)^2,$$

$$M = -\frac{2716}{89 \times 100}\left(\frac{11}{25}\right)^2,\ N = \frac{1224}{41 \times 100}\left(\frac{11}{25}\right)^2,\ M - 2e(k^2 - c^2 z) = -\frac{4080}{89 \times 100}\left(\frac{11}{25}\right)^2.$$

These values reduce the equation (22) to the identity

$$\frac{89}{41} = \frac{89}{41}\left\{\frac{41(4080^2 + 5 \times 1224^2) - 89\ (2448 \times 4080)}{89(4080^2 + 5 \times 1224^2) - 205\ (2448 \times 4080)}\right\}.$$

§9. *Example Second.* The example that has been given is one in which the auxiliary biquadratic is irreducible. I will now take an example,

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0, \tag{23}$$

in which the auxiliary biquadratic has a sub-auxiliary quadratic. When the root of the equation (23) is put in the form (1),

$$u_1 = (1 + \sqrt{2})^{\frac{1}{5}},\ u_4 = (1 - \sqrt{2})^{\frac{1}{5}},\ u_2 = (1 + \sqrt{2})(1 + \sqrt{2})^{\frac{2}{5}},\ u_3 = (1 - \sqrt{2})(1 - \sqrt{2})^{\frac{2}{5}},$$

the product of the roots $(1 + \sqrt{2})^{\frac{1}{5}}$, $(1 - \sqrt{2})^{\frac{1}{5}}$, being -1. Putting β for 28560, and λ for 28562,

$$g = -1,\ k = 4,\ c\sqrt{z} = -3,\ z = \frac{\lambda^2}{\beta^2},\ c = \frac{3\beta}{\lambda},\ k^2 + c^2 z = 25,\ kc = \frac{12\beta}{\lambda},\ kcz = \frac{12\lambda}{\beta},$$

$$P = 2(k^2 + c^2 z) - g^3 = 51,\ M = \frac{48\lambda - 51\beta}{\beta},\ N = \frac{51\lambda - 48\beta}{\lambda},$$

$$M - 2e(k^2 - c^2 z) = \frac{48\lambda - 51\beta + 14 \times 338}{\beta}.$$

These values cause (22) to become

$$\frac{13}{12} = \frac{\lambda\{(48\lambda - 51\beta + 14 \times 338)^2 + (51\lambda - 48\beta)^2\} + 2\beta(51\lambda - 48\beta)(48\lambda - 51\beta + 14 \times 338)}{\beta\{(48\lambda - 51\beta + 14 \times 338)^2 + (51\lambda - 48\beta)^2\} + 2\lambda(51\lambda - 48\beta)(48\lambda - 51\beta + 14 \times 338)}.$$

This may be written $\frac{13}{12} = \frac{H\lambda + 2K\beta}{H\beta + 2K\lambda}$. In order that this equation may subsist, it is necessary that

$$H(13\beta - 12\lambda) = 2K(12\beta - 13\lambda);\ \text{or}\ \frac{H}{2}\left(\frac{\beta - 24}{2}\right) = -\frac{K(\beta + 26)}{2}.$$

But $H = (-80852)^2 + (85782)^2 = 6537045904 + 7358551524 = 13895597428$;

$-K = (80852)(85782) = 6935646264$; $\dfrac{\beta - 24}{2} = 14268$; $\dfrac{\beta + 26}{2} = 14293$;

and $6947798714 \times 14268 = 6935646264 \times 14293 = 99131192051352$.

§10. *Example Third.* I will finally take an example,

$$x^5 + 20x^3 + 20x^2 + 30x + 10 = 0, \tag{24}$$

in which the roots of the auxiliary biquadratic are all rational. By (6) and (10) and (16), $g = -2$, $k = -1$, $c^2 z = 0$. Therefore the denominator of the expression on the left of (22) is zero, while the numerator is not zero. Therefore the denominator of the expression on the right of (22) is zero. Or, $-g^6 + 4k^2 g^3 - 8ek^4 + 4eg^3 k^2 = 0$. Therefore $e = -\frac{12}{5}$. Therefore $z = \left(\frac{13}{5}\right)^2$, and $c = 0$. Hence $M = -10$, $N = 10$; and, if

$$D = M^2 - zN^2 - 4e^2 (k^2 - c^2 z)^2,$$

$D = -104e^2$. Therefore, by (21), $\theta = -\frac{1}{12}$, $\phi = \frac{25}{12 \times 13}$, $\theta^2 - \phi^2 z = -\frac{1}{6}$. Therefore, by (9), $h = \frac{225}{26}$. Therefore, using the symbols, B, B', as in §3,

$$B = -\frac{5}{2},\ B' = -\frac{45}{26},\ s = h(z + \sqrt{z}) = 81,\ s_1 = h(z - \sqrt{z}) = 36.$$

Therefore $u_1^5 = -7 + 9 = 2$, $u_4^5 = -7 - 9 = -16$, $u_2^5 = 2 - 6 = -4$, $u_3^5 = 2 + 6 = 8$. Hence, by (3),

$$r_1 = 2^{\frac{1}{5}} - 2^{\frac{2}{5}} + 2^{\frac{3}{5}} - 2^{\frac{4}{5}};$$

which is the solution of the equation (24).

§11. It was pointed out in §7 that, in the case we are considering, there are six equations and five unknown quantities. All the unknown quantities may be eliminated, and an equation $p' = 0$ obtained; where p' is a rational function of the coefficients p_2, p_3, etc. This elimination has been performed, under the direction of the author of the paper, by Mr. Warren Reid of Toronto, with the following result. Putting P, as in §7, for $2(k^2 + c^2 z) - g^3$, let

$$A = -2kc^2 zg^3 \{8(k^2 + c^2 z) - 3g^3\},$$
$$B = g^3\{16k^2 c^2 z + 4(k^2 + c^2 z)^2 - 5g^3 (k^2 + c^2 z) + g^6\},$$
$$D = -4(k^2 - c^2 z)\{-g^6 + 3g^3 (k^2 + c^2 z) - 2(k^2 - c^2 z)^2\},$$
$$A_1 = -8kc^2 z[32kc^2 z(k^2 - c^2 z) - P\{p_5 g^2 + 8k(k^2 - c^2 z) - 4kg^3\}]$$
$$B_1 = \{p_5 g^2 + 8k(k^2 - c^2 z) - 4kg^3\}[-32k^2 c^2 z + g^3\{4(k^2 + c^2 z) - g^3\}] + 64kc^2 z P(k^2 - c^2 z),$$
$$D_1 = -16kc^2 zg^3\{4(k^2 + c^2 z) - g^3\} + 4P(k^2 - c^2 z)\{p_5 g^2 + 8k(k^2 - c^2 z) - 4kg^3\}.$$

Then, since $10g = -p_2$, and $20k = -p_3$, and $20c^2z = p_4g - 5g^3 + 20k^2$, the quantities A, B, D, A_1, B_1, D_1, are known rational functions of p_2, p_3, etc. And

$$(B^2 + D^2)(A_1^2 - D_1^2c^2z) - (B_1^2 + D_1^2)(A^2 - D^2c^2z)$$
$$+ 4\{AB(B_1^2 + D_1^2) - A_1B_1(B^2 + D^2)\}\{AB(A_1^2 - D_1^2c^2z) - A_1B_1(A^2 - D^2c^2z)\} = 0. \quad (25)$$

§12. To verify this result, the Gaussian equation in §8 may be used. Here

$$A = -\frac{11^6}{2^5 \times 5^{12}}\left(\frac{11^3 + 11^2 \times 19}{5^6}\right) = -\frac{11^8 \times 3}{2^4 \times 5^{17}}$$
$$B = \frac{11^3}{5^6}\left(\frac{11^4}{2^4 \times 5^9} + \frac{3^4 \times 7^2 \times 11^4}{2^4 \times 5^{12}} - \frac{9 \times 35 \times 11^5}{8 \times 5^{12}} + \frac{11^6}{5^{12}}\right) = -\frac{9 \times 11^7}{4 \times 5^{16}}$$
$$D = \frac{11^2 \times 31}{5^{18}}\left(-11^6 + \frac{7 \times 27 \times 11^5}{8} - \frac{31^2 \times 11^4}{8}\right) = \frac{3 \times 31 \times 11^6}{4 \times 5^{16}}$$
$$A_1 = \frac{11^8}{2^6 \times 5^{18}}(19 + 31) = \frac{11^8}{2^5 \times 5^{16}}$$
$$B_1 = \frac{11^7}{2^4 \times 5^{18}}(-5^3 + 44 \times 41 - 19 \times 31) = \frac{11^7 \times 109}{8 \times 5^{17}}$$
$$D_1 = -\frac{11^6}{4 \times 5^{12}}\left(\frac{63 \times 11^2}{2 \times 5^6} - \frac{11^3}{5^6}\right) - \frac{11^7 \times 19 \times 31}{8 \times 5^{18}} = -\frac{11^7 \times 26}{5^{17}}.$$

Therefore $B^2 + D^2 = \dfrac{9 \times 11^{12} \times 41}{8 \times 5^{30}}$, $B_1^2 + D_1^2 = \dfrac{11^4 \times 11029}{2^6 \times 5^{33}}$,

$$A^2 - D^2c^2z = -\frac{9 \times 11^{14} \times 89}{2^6 \times 5^{35}}, \quad A_1^2 - D_1^2c^2z = -\frac{11^{16} \times 40139}{2^{10} \times 5^{37}}.$$

By the substitution of these values, equation (25) becomes

$$\frac{11^{56} \times 3^4}{2^{26} \times 5^{136}}\{6265333^2 - 2886277 \times 13600357\} =$$
$$\frac{11^{56} \times 3^4}{2^{26} \times 5^{136}}\{39254397600889 - 39254397600889\} = 0.$$

§13. As an additional verification, the equation

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0$$

may be taken. Here, by §9, $g = -1$, $k = 4$, $k^2 - c^2z = 7$, $k^2 + c^2z = 25$. Therefore

$$A = 2^3 \times 3^2 \times 7 \times 29, \quad B = -2 \times 5 \times 17 \times 29, \quad D = 2^3 \times 3 \times 7 \times 29,$$
$$A_1 = -2^9 \times 3^4 \times 141, \quad B_1 = 2^4 \times 3 \times 17 \times 2393, \quad D_1 = -2^7 \times 3^2 \times 13 \times 19.$$
$$B^2 + D^2 = 2^2 \times 29^2 \times 14281, \quad B_1^2 + D_1^2 = 2^8 \times 3^2 \times 5 \times 338016989,$$
$$A^2 - D^2c^2z = 0, \quad A_1^2 - D_1^2c^2z = 2^{14} \times 3^8 \times 5 \times 7 \times 17^2 \times 277.$$

By the substitution of these values, equation (25) becomes

$$2^{18} \times 3^6 \times 5 \times 7 \times 17^2 \times 29^4\{277 \times 14281^2$$
$$+ 5^2 \times 7 \times 338016989 - 2^3 \times 3 \times 141 \times 2393 \times 14281\} = 0.$$

The Trinomial Quintic $x^5+p_4x+p_5=0$.

§14. In this case, by (6) and (10), $g=0$, and $k=0$. Therefore, by (11), $A=-\frac{he(\theta^2-\varphi^2z)}{a}$. Therefore, by (12),

$$p_4=\frac{20he(\theta^2-\varphi^2z)}{a}+15a^2z. \tag{26}$$

Also, by §3, $B=\frac{1}{a^2z}\{-a^3z^2c+2hecz(\theta^2-\phi^2z)\}$. Therefore, by (13),

$$p_5=-\frac{8hec}{a^2}(\theta^2-\phi^2z)+44acz. \tag{27}$$

Hence the quintic becomes

$$F(x)=x^5+\left\{\frac{20he(\theta^2-\varphi^2z)}{a}+15a^2z\right\}x+\left\{-\frac{8hec}{a^2}(\theta^2-\phi^2z)+44acz\right\}=0. \tag{28}$$

The criterion of solvability of a trinomial quintic of the kind under consideration is therefore that the coefficients p_4 and p_5 be related in the manner indicated in the form (28); while at the same time the last four of equations (15), modified by putting $g=k=0$, subsist between the rational quantities a, c, e, h, θ, ϕ. From these data, the three following equations may be deduced, v being put for $\frac{c^2}{a^3}$:

$$\left.\begin{aligned}&8ev^3-4zv^2+z(3-4e)v-z^2=0,\\&\frac{2p_4}{a^2}+\frac{5p_5}{ac}=250z,\\&4v(ze+4zv-8v^2)=\left(-3z+\frac{p_4}{5a^2}\right)\{z+4v(e-1)+8v^2\}.\end{aligned}\right\} \tag{29}$$

The first of these equations is obtained from a comparison of the two equations (9); the second is obtained by putting p_4 and p_5 respectively equal to the values they have in (28); and the third is obtained by putting p_4 equal to the coefficient of the first power of x in (28).

§15. If any rational values of e and v can be found satisfying the first of equations (29), let such values be taken. Then, from the second and third of (29), a^2 and ac can be found. Therefore a and c are known. Therefore, by (21), θ and ϕ are known. Therefore, by (9), h is known. In this way all the elements for the solution of the quintic are obtained.

§16. For example, the three equations (29) are satisfied by the values,

$$e=\frac{1}{2},\ z=v=\frac{5}{4},\ c=\frac{25}{2},\ a=5,\ \therefore\ \theta=0,\ \phi=-\frac{4}{75},\ h=\frac{45\times25^3}{16}.$$

When these values are substituted in (28), the quintic becomes

$$x^5+\frac{625x}{4}+3750=0.$$

Then the values of u_1^5, u_2^5, u_3^5, u_4^5, obtained from the expression for u_1^5 in §3, are

$$u_1^5=\frac{625}{4}\left\{-1-\sqrt{\left(\frac{5}{4}\right)}+\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}+\sqrt{\frac{5}{4}}\right)}\right\},$$

$$u_4^5=\frac{625}{4}\left\{-1-\sqrt{\left(\frac{5}{4}\right)}-\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}+\sqrt{\frac{5}{4}}\right)}\right\},$$

$$u_2^5=\frac{625}{4}\left\{-1+\sqrt{\left(\frac{5}{4}\right)}-\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}-\sqrt{\frac{5}{4}}\right)}\right\},$$

$$u_3^5=\frac{625}{4}\left\{-1+\sqrt{\left(\frac{5}{4}\right)}+\frac{3}{\sqrt{5}}\sqrt{\left(\frac{5}{4}-\sqrt{\frac{5}{4}}\right)}\right\}.$$

Hence, $r_1=u_1+u_2+u_3+u_4=-1.52887-2.25035+2.48413-3.65639=-4.95148$.

WHEN ANY RELATION IS ASSUMED BETWEEN THE SIX UNKNOWN QUANTITIES.

§17. In the case in which u_1u_4 was taken equal to u_2u_3, a relation was in fact assumed betwixt the six unknown quantities a, c, e, h, θ, ϕ; for, as we saw, to put $u_1u_4=u_2u_3$ is tantamount to putting $a=0$. Hence, as was noticed in §7, we had only five unknown quantities to be found from six equations. Now, when any relation whatever is assumed betwixt the six unknown quantities, the root of the quintic can be found in terms of the given coefficients p_2, p_3, etc., without any definite numerical values being assigned to the coefficients, because six rational quantities can always be found from seven equations.

THE GENERAL CASE.

§18. We have hitherto been dealing with solvable quintics, assumed to be subject to some condition additional to what is involved in their solvability. We have now to consider how the general case is to be dealt with. That is to say, we here make no supposition regarding the equation of the fifth degree $F(x)=0$ except that it wants the second term and is solvable algebraically. In this case it is impossible to find the roots in terms of the coefficients p_2, p_3, etc., while these coefficients retain their general symbolic forms. But the equations in §3 enable us to find the roots when the coefficients receive any definite numerical values that render the equation solvable. For, we have the six equations (15) to determine the six unknown quantities a, c, e, h, θ, ϕ; and we

can eliminate five of the unknown quantities, and obtain an equation involving only one unknown quantity. The unknown quantity appearing in this equation has a rational value; but there are known methods of finding the rational roots of any algebraical equation with definite numerical coefficients. Therefore the unknown quantity can be found. In this way all the six unknown quantities a, c, e, h, θ, ϕ, can be found. Hence the roots of the quintic can be found.

§19. *Note.*—From my friend, Mr. J. C. Glashan, of Ottawa, who read in manuscript the paper on the "*Principles of the Solution of Equations of the Higher Degrees,*" but did not see the present paper on the "*Resolution of Solvable Equations of the Fifth Degree,*" I learn that, setting out from propositions demonstrated in the "Principles," he has arrived at important conclusions in the theory of Quintics, which will be made public without delay; but he has not communicated to me either his method or the results he has obtained.

On Certain Possible Abbreviations in the Computation of the Long-Period Inequalities of the Moon's Motion due to the Direct Action of the Planets.

By G. W. Hill.

Hansen has characterized the calculation of the coefficients of these inequalities as extremely difficult. However, it seems to me that, if the shortest methods are followed, there is no ground for such an assertion. The work may be divided into two portions, independent of each other. In one the object is to develop, in periodic series, certain functions of the moon's coordinates, which in number do not exceed five. This portion is the same whatever planet may be considered to act, and hence may be done once for all. In the other portion we seek the coefficients of certain terms in the periodic development of certain functions, five also in number, which involve the coordinates of the earth and planet only. And this part of the work is very similar to that in which the perturbations of the earth by the planet in question are the things sought. And as the multiples of the mean motions of these two bodies, which enter into the expression of the argument of the inequalities under consideration, are necessarily quite large, approximative values of the coefficients may be obtained by semi-convergent series similar to the well-known theorem of Stirling. This matter was first elaborated by Cauchy,* but, in the method as left by him, we are directed to compute special values of the successive derivatives of the functions to be developed. Now it unfortunately happens that these functions are enormously complicated by successive differentiation, so that it is almost impossible to write at length their second derivatives. Manifestly then it would

* *Mémoire sur les approximations des fonctions de très-grands nombres;* and *Rapport sur un Mémoire de M. Le Verrier, qui a pour objet la détermination d'une grande inégalité du moyen mouvement de la planète Pallas: Comptes Rendus de l'Académie des Sciences de Paris, Tom.* XX, *pp.* 691–726, 767–786, 825–847.

be a great saving of labor to substitute for the computation of special values of these derivatives a computation of a certain number of special values of the original function, distributed in such a way that the maximum advantage may be obtained. This modification has given rise to an elegant piece of analysis. It will be noticed that, in this method, it is necessary to substitute in the formulæ, from the outset, the numerical values of the elements of the orbits of the earth and planet. There seems to be no objection to this on the practical side, as, for the computation of the inequalities sought, no partial derivatives of R, with respect to these elements, are required.

I.

If the masses of the moon, earth and the planet considered are denoted severally by m, M and m'', and the geocentric rectangular coordinates of the moon by x, y, and z, the similar coordinates of the sun by x', y' and z', and the heliocentric coordinates of the planet by x'', y'' and z'', the perturbative function, for the direct action of the planet on the moon, is

$$R = m''\left[\frac{1}{[(x''+x'-x)^2+(y''+y'-y)^2+(z''+z'-z)^2]^{\frac{1}{2}}} - \frac{(x''+x')x+(y''+y')y+(z''+z')z}{[(x''+x')^2+(y''+y')^2+(z''+z')^2]^{\frac{3}{2}}}\right].$$

But, by a slight substitution in and modification of this expression, we take account of the lunar perturbations of the solar coordinates. Let X, Y and Z denote the coordinates of the sun referred to the centre of gravity of the earth and moon, we shall then have

$$x' = X + \frac{m}{M+m}x, \quad y' = Y + \frac{m}{M+m}y, \quad z' = Z + \frac{m}{M+m}z.$$

And Δ may denote the distance of the planet from the centre of gravity of the earth and moon, so that

$$\Delta^2 = (x''+X)^2 + (y''+Y)^2 + (z''+Z)^2,$$

also r the radius vector of the moon, so that

$$r^2 = x^2 + y^2 + z^2;$$

moreover, for brevity, put

$$P = (x''+X)x + (y''+Y)y + (z''+Z)z.$$

Then R takes the form

$$R = m''\left[\frac{1}{\left[\Delta^2 - 2\frac{M}{M+m}P + \frac{M^2}{(M+m)^2}r^2\right]^{\frac{1}{2}}} - \frac{P + \frac{m}{M+m}r^2}{\left[\Delta^2 + 2\frac{m}{M+m}P + \frac{m^2}{(M+m)^2}r^2\right]^{\frac{3}{2}}}\right].$$

But it is evident that this expression, differentiated with respect to the variables x, y and z, will not furnish differential coefficients identical in value with those the expression gives before the transformation, as x', y' and z' have now been made to involve x, y and z. But a little consideration shows the modification which will remedy this. It is plain we ought to multiply the first term by $\frac{M+m}{M}$, and, multiplying the last term by $-\frac{M+m}{m}$, substitute unity for the numerator and reduce the exponent of the denominator from $\frac{3}{2}$ to $\frac{1}{2}$.

Thus the proper form of R is

$$R = m''\left[\frac{M+m}{M}\frac{1}{\left[\Delta^2 - 2\frac{M}{M+m}P + \frac{M^2}{(M+m)^2}r^2\right]^{\frac{1}{2}}} + \frac{M+m}{m}\frac{1}{\left[\Delta^2 + 2\frac{m}{M+m}P + \frac{m^2}{(M+m)^2}r^2\right]^{\frac{1}{2}}}\right].$$

When this expression is expanded in a series proceeding according to ascending powers of the lunar coordinates, and the terms independent of the latter omitted, we get

$$\begin{aligned} R = m''\Big\{ & \frac{4.3}{2.4}\frac{P^2}{\Delta^5} - \frac{2}{1}\cdot\frac{2.1}{2.4}\frac{r^2}{\Delta^3} \\ & + \frac{M^2 - m^2}{(M+m)^2}\left[\frac{6.5.4}{2.4.6}\frac{P^3}{\Delta^7} - \frac{3}{1}\cdot\frac{4.3.2}{2.4.6}\frac{Pr^2}{\Delta^5}\right] \\ & + \frac{M^3 + m^3}{(M+m)^3}\left[\frac{8.7.6.5}{2.4.6.8}\frac{P^4}{\Delta^9} - \frac{4}{1}\cdot\frac{6.5.4.3}{2.4.6.8}\frac{P^2r^2}{\Delta^7} + \frac{4.3}{1.2}\cdot\frac{4.3.2.1}{2.4.6.8}\frac{r^4}{\Delta^5}\right] \\ & + \ldots\ldots\ldots\ldots \Big\}. \end{aligned}$$

The terms of this series follow a quite evident law, and it is easy to write as many as there may be occasion for. But, hitherto, no sensible inequalities have been found arising from the terms beyond the first line. This line furnishes all the inequalities which are not factored by the small ratio $\frac{a}{a'}$, whose value is about $\frac{1}{400}$. And the following two lines of terms can add to the coefficients of these only parts which have the very small factor $\frac{a^2}{a'^2}$. For these reasons we can restrict ourselves to the first line of terms, and write very simply

$$R = m''\left[\frac{3}{2}\frac{P^2}{\Delta^5} - \frac{1}{2}\frac{r^2}{\Delta^3}\right].$$

Restoring the equivalent of P,

$$R = m''\Big\{\Big[\frac{3}{2}\frac{(x''+X)^2}{\varDelta^5}-\frac{1}{2}\frac{1}{\varDelta^3}\Big]x^2+\Big[\frac{3}{2}\frac{(y''+Y)^2}{\varDelta^5}-\frac{1}{2}\frac{1}{\varDelta^3}\Big]y^2$$
$$+\Big[\frac{3}{2}\frac{(z''+Z)^2}{\varDelta^5}-\frac{1}{2}\frac{1}{\varDelta^3}\Big]z^2+3\frac{(x''+X)(y''+Y)}{\varDelta^5}xy$$
$$+3\frac{(x''+X)(z''+Z)}{\varDelta^5}xz+3\frac{(y''+Y)(z''+Z)}{\varDelta^5}yz\Big\}.$$

This expression has the advantage of exhibiting the value of R as a sum of terms of which each is the product of two factors, one of which depends solely on the coordinates of the moon and the other is independent of them.

If we denote the factors of x^2, y^2 and z^2 in R severally by A, B and C, we shall have the relation $\quad A+B+C=0$.
Hence it is plain that the number of terms can be reduced from six to five. As we shall take the ecliptic for the plane of xy, we will have $Z=0$. We can then write

$$R = m''\Big\{\frac{1}{4}\Big[\frac{1}{\varDelta^3}-3\frac{z''^2}{\varDelta^5}\Big](r^2-3z^2)+\frac{3}{4}\frac{(y''+Y)^2-(x''+X)^2}{\varDelta^5}(y^2-x^2)$$
$$+3\frac{(x''+X)(y''+Y)}{\varDelta^5}xy+3\frac{(x''+X)z''}{\varDelta^5}xz+3\frac{(y''+Y)z''}{\varDelta^5}yz\Big\}.$$

II.

We will now express the five factors of the terms of R, viz. r^2-3z^2, x^2-y^2, xy, xz, and yz, as functions of t, the time, when elliptic values are attributed to the coordinates, leaving, however, the longitudes of the perigee and node indeterminate, so that the latter may have their motions proportional to t.

Using Delaunay's notation, and, in addition, putting v for the true anomaly, we have

$$x = r\cos(v+g)\cos h-(1-2\gamma^2)\,r\sin(v+g)\sin h,$$
$$y = r\cos(v+g)\sin h+(1-2\gamma^2)\,r\sin(v+g)\cos h,$$
$$z = 2\gamma\sqrt{1-\gamma^2}\,r\sin(v+g);$$

or, in a slightly different form,

$$x = (1-\gamma^2)\,r\cos(v+g+h)+\gamma^2 r\cos(v+g-h),$$
$$y = (1-\gamma^2)\,r\sin(v+g+h)-\gamma^2 r\sin(v+g-h),$$
$$z = 2\gamma\sqrt{1-\gamma^2}\,r\sin(v+g).$$

From these equations we derive

$$z^2 = 2\gamma^2(1-\gamma^2)r^2[1-\cos 2(v+g)],$$
$$r^2-3z^2 = [1-6\gamma^2+6\gamma^4]\,r^2+6\gamma^2(1-\gamma^2)r^2\cos 2(v+g),$$
$$x^2-y^2 = (1-\gamma^2)^2r^2\cos 2(v+g+h)+\gamma^4r^2\cos 2(v+g-h)+2\gamma^2(1-\gamma^2)r^2\cos 2h,$$

$$2xy = (1-\gamma^2)^2 r^2 \sin 2(v+g+h) - \gamma^4 r^2 \sin 2(v+g-h) + 2\gamma^2(1-\gamma^2) r^2 \sin 2h,$$
$$xz = \gamma(1-\gamma^2)^{\frac{3}{2}} r^2 \sin(2v+2g+h) + \gamma^3(1-\gamma^2)^{\frac{1}{2}} r^2 \sin(2v+2g-h)$$
$$- \gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}} r^2 \sin h,$$
$$yz = -\gamma(1-\gamma^2)^{\frac{3}{2}} r^2 \cos(2v+2g+h) + \gamma^3(1-\gamma^2)^{\frac{1}{2}} r^2 \cos(2v+2g-h)$$
$$+ \gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}} r^2 \cos h.$$

It is then plain that the development of these five factors depends on that of the quantities r^2, $r^2 \cos 2v$ and $r^2 \sin 2v$. Denoting the eccentric anomaly by u, we have

$$\frac{r^2}{a^2} = (1-e\cos u)^2,$$
$$\frac{r^2}{a^2}\cos 2v = \frac{3}{2}e^2 - 2e\cos u + \left(1-\frac{1}{2}e^2\right)\cos 2u,$$
$$\frac{r^2}{a^2}\sin 2v = \sqrt{1-e^2}(\sin 2u - 2e\sin u).$$

The constant terms of these functions, in their development in periodic series involving multiples of the mean anomaly, are the same as the constant terms of the right members of the last equations after they have been multiplied by $1-e\cos u$. That is, these terms are severally $1+\frac{3}{2}e^2$, $\frac{5}{2}e^2$ and 0. To obtain the remaining coefficients, we put $s = \varepsilon^{u\sqrt{-1}}$, and $z = \varepsilon^{l\sqrt{-1}}$, and recall the theorem that the coefficient of z^i, in the development of any function S according to powers of z, is the same as that of s^i in the development of

$$\frac{s}{i}\frac{dS}{ds}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)},$$

according to powers of s. Moreover, adopting Hansen's notation for the Besselian function, we put $\varepsilon^{\lambda\left(s-\frac{1}{s}\right)} = \Sigma_i . J_\lambda^{(i)} s^i,$

so that, for positive values of i, we have

$$J_\lambda^{(i)} = \frac{\lambda^i}{1.2\ldots i}\left[1 - \frac{\lambda^2}{1.(i+1)} + \frac{\lambda^4}{1.2(i+1)(i+2)} - \ldots\right],$$

and, for negative values, $J_\lambda^{(-i)} = J_{-\lambda}^{(i)}$.

These functions satisfy the following equation,

$$iJ_\lambda^{(i)} = \lambda\left(J_\lambda^{(i-1)} + J_\lambda^{(i+1)}\right).$$

Whence

$$J_\lambda^{(i-1)} = \frac{i}{\lambda}J_\lambda^{(i)} - J_\lambda^{(i+1)},$$
$$J_\lambda^{(i+1)} = \frac{i}{\lambda}J_\lambda^{(i)} - J_\lambda^{(i-1)},$$

and, by writing $i-1$ for i in the first of these and $i+1$ for i in the second,

$$J_\lambda^{(i-2)} = \frac{i-1}{\lambda}J_\lambda^{(i-1)} - J_\lambda^{(i)},$$
$$J_\lambda^{(i+2)} = \frac{i+1}{\lambda}J_\lambda^{(i+1)} - J_\lambda^{(i)}.$$

Consequently $$J_\lambda^{(i-2)} - J_\lambda^{(i+2)} = \frac{1}{\lambda}\left[(i-1)J_\lambda^{(i-1)} - (i+1)J_\lambda^{(i+1)}\right].$$

The coefficient of z^i in the expansion of $\frac{r^2}{a^2}$ being equal to that of s^i in

$$-\frac{e}{i}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]\left(s-\frac{1}{s}\right)\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)},$$

is $$-\frac{e}{i}\left[J_{\frac{ie}{2}}^{(i-1)} - J_{\frac{ie}{2}}^{(i+1)} - \frac{e}{2}\left(J_{\frac{ie}{2}}^{(i-2)} - J_{\frac{ie}{2}}^{(i+2)}\right)\right],$$

which, by means of the relations between the J functions just given, reduces to

$$-\frac{2}{i^2}J_{\frac{ie}{2}}^{(i)}.$$

Hence we have $$\frac{r^2}{a^2} = 1 + \frac{3}{2}e^2 - \sum_{i=1}^{i=\infty} . \frac{4}{i^2}J_{\frac{ie}{2}}^{(i)} \cos il.$$

This result may also be obtained from the equation

$$\frac{d^2\frac{r^2}{a^2}}{dl^2} = 2\frac{a}{r} - 2.$$

In like manner we get

$$\frac{r^2}{a^2}\cos 2v = \frac{5}{2}e^2 + \sum_{i=1}^{i=\infty} . \frac{2}{i}\left[\left(1-\frac{1}{2}e^2\right)\left(J_{\frac{ie}{2}}^{(i-2)} - J_{\frac{ie}{2}}^{(i+2)}\right) - e\left(J_{\frac{ie}{2}}^{(i-1)} - J_{\frac{ie}{2}}^{(i+1)}\right)\right]\cos il,$$

$$\frac{r^2}{a^2}\sin 2v = \sqrt{1-e^2}\sum_{i=1}^{i=\infty} . \frac{2}{i}\left[J_{\frac{ie}{2}}^{(i-2)} + J_{\frac{ie}{2}}^{(i+2)} - e\left(J_{\frac{ie}{2}}^{(i-1)} + J_{\frac{ie}{2}}^{(i+1)}\right)\right]\sin il.$$

Consequently, if we put

$$H^{(i)} = \frac{2}{i}\left[\left(\cos^2\frac{\varphi}{2} - \frac{1}{4}e^2\right)J_{\frac{ie}{2}}^{(i-2)} - e\cos^2\frac{\varphi}{2} \cdot J_{\frac{ie}{2}}^{(i-1)}\right.$$
$$\left. + e\sin^2\frac{\varphi}{2} . J_{\frac{ie}{2}}^{(i+1)} - \left(\sin^2\frac{\varphi}{2} - \frac{1}{4}e^2\right)J_{\frac{ie}{2}}^{(i+2)}\right],$$

where $\sin\phi = e$, and we agree that

$$H^{(0)} = \frac{5}{2}e^2,$$

we shall have, α denoting any arbitrary angle,

$$r^2\cos(\alpha + 2v) = a^2\sum_{i=-\infty}^{i=+\infty} . H^{(i)}\cos(\alpha + il),$$

$$r^2\sin(\alpha + 2v) = a^2\sum_{i=-\infty}^{i=+\infty} . H^{(i)}\sin(\alpha + il).$$

We can now write the expansions of the five factors of the terms of R which depend solely on the moon's coordinates:

$$\frac{r^2-3z^2}{4a^2} = -\frac{1}{2}(1-6\gamma^2+6\gamma^4)\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\cos il$$
$$+\frac{3}{2}\gamma^2(1-\gamma^2)\Sigma.H^{(i)}\cos(2g+il),$$
$$\frac{3}{4}\frac{x^2-y^2}{a^2} = \frac{3}{4}(1-\gamma^2)^2\Sigma.H^{(i)}\cos(2h+2g+il)$$
$$-3\gamma^2(1-\gamma^2)\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\cos(2h+il)$$
$$+\frac{3}{4}\gamma^4\Sigma.H^{(i)}\cos(-2h+2g+il),$$
$$\frac{3}{2}\frac{xy}{a^2} = \frac{3}{4}(1-\gamma^2)^2\Sigma.H^{(i)}\sin(2h+2g+il)$$
$$-3\gamma^2(1-\gamma^2)\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\sin(2h+il)$$
$$-\frac{3}{4}\gamma^4\Sigma.H^{(i)}\sin(-2h+2g+il),$$
$$\frac{3}{2}\frac{xz}{a^2} = \frac{3}{2}\gamma(1-\gamma^2)^{\frac{3}{2}}\Sigma.H^{(i)}\sin(h+2g+il)$$
$$+3\gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}}\Sigma.\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}\sin(h+il)$$
$$+\frac{3}{2}\gamma^3(1-\gamma^2)^{\frac{1}{2}}\Sigma.H^{(i)}\sin(-h+2g+il),$$
$$\frac{3}{2}\frac{yz}{a^2} = -\frac{3}{2}\gamma(1-\gamma^2)^{\frac{3}{2}}\Sigma.H^{(i)}\cos(h+2g+il)$$
$$-3\gamma(1-2\gamma^2)(1-\gamma^2)^{\frac{1}{2}}\Sigma.\frac{1}{i}J^{(i)}_{\frac{ie}{2}}\cos(h+il)$$
$$+\frac{3}{2}\gamma^3(1-\gamma^2)^{\frac{1}{2}}\Sigma.H^{(i)}\cos(-h+2g+il).$$

The summation must be extended to all integral values positive and negative, zero included, for i. When $i=0$ we must suppose that $\frac{1}{i^2}J^{(i)}_{\frac{ie}{2}}$ takes the value $-\frac{1}{2}\left(1+\frac{3}{2}e^2\right)$.

It will be perceived that the three first terms of R furnish inequalities whose arguments do not involve the longitude of the moon's node or involve it in an even multiple. The two remaining terms furnish inequalities having an odd multiple of this longitude in their arguments. And it is evident that these statements remain true even when the solar perturbations of the lunar coordinates are taken into consideration. Hence, in deriving any particular inequality, we never have to consider more than three out of the five terms of R. When we propose to neglect the solar perturbations, it can be seen at a glance what terms of the expressions above ought to be retained. Thus, in the case of

Hansen's inequality of 273 years, the argument involving only l without either h or g, it is plain that the first term of $\frac{r^2-3z^2}{4a^2}$ can alone furnish it; and, consequently, we may put, very simply,

$$R=-m''a^2(1-6\gamma^2+6\gamma^4)J'_{\frac{e}{2}}\left[\frac{1}{\varDelta^3}-3\frac{z'^2}{\varDelta^5}\right]\cos l.$$

And the whole difficulty is reduced to finding, in the development of

$$\frac{1}{\varDelta^3}-3\frac{z'^2}{\varDelta^5},$$

the terms $$A^{(c)}\cos(18l''-16l')+A^{(s)}\sin(18l''-16l').$$

III.

We pass now to the consideration of the development, in periodic series, of the factors of the terms of R which depend on the coordinates of the earth and planet. Let it be required to discover the coefficient $C_{i,i'}$ of $z^i z'^{i'}$ in the development of any periodic function of the eccentric anomalies u and u' of two planets, in the case where i is quite large. We shall suppose that the function has $\frac{1}{\varDelta^{2n}}$ for a factor. It is known that

$$\frac{1}{\varDelta^{2n}}=N^{2n}[1-2\mathfrak{a}\cos(u-Q)+\mathfrak{a}^2]^{-n}[1-2\mathfrak{b}\cos(u+Q)+\mathfrak{b}^2]^{-n},$$

where N, $\mathfrak{a}$, $\mathfrak{b}$ and Q are functions of u' or l' only, and $\mathfrak{a}$ and $\mathfrak{b}$ are always less than unity. Substituting the imaginary exponential $s=\varepsilon^{u\sqrt{-1}}$, and, to abbreviate, putting $$k=\mathfrak{a}^{-1}\varepsilon^{Q\sqrt{-1}},\quad k_1=\mathfrak{b}^{-1}\varepsilon^{-Q\sqrt{-1}},$$
this equation becomes

$$\frac{1}{\varDelta^{2n}}=N^{2n}\left(1-\frac{s}{k}\right)^{-n}\left(1-\frac{\mathfrak{a}^2k}{s}\right)^{-n}\left(1-\frac{s}{k_1}\right)^{-n}\left(1-\frac{\mathfrak{b}^2k_1}{s}\right)^{-n}.$$

Rendering evident the factor $\left(1-\frac{s}{k}\right)^{-n}$, we can then suppose that the function to be developed is

$$\left(1-\frac{s}{k}\right)^{-n}F(s).$$

The coefficient of z^i in the development of this is equivalent to

$$C_i=\frac{1}{2\pi}\int_0^{2\pi}s^{-i}\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]\left(1-\frac{s}{k}\right)^{-n}F(s)\,du.$$

Let us put $$f(s)=\varepsilon^{\frac{ie}{2}\left(s-\frac{1}{s}\right)}\left[1-\frac{e}{2}\left(s+\frac{1}{s}\right)\right]F(s);$$

then $$C_i=\frac{1}{2\pi}\int_0^{2\pi}s^{-i}\left(1-\frac{s}{k}\right)^{-n}f(s)\,du.$$

Since the absolute term of a series of integral powers of a variable is not

changed by substituting for the latter a constant multiple of it, in the expression for C_i we can write ks for s. Thus

$$C_i = \frac{k^{-i}}{2\pi}\int_0^{2\pi} s^{-i}(1-s)^{-n} f(ks)\, du.$$

The difficulty here that the factor $(1-s)^{-n}$ becomes infinite at the limits of the definite integral, is only apparent. For the multiple of s instead of ks may be ps, in which the modulus of p is less than that of k by a very small quantity. In this case we get a tangible result, which is seen to have, as its limit, when p is made to approach k indefinitely, the value which will be presently given.

We now assume that it is possible to expand $f(ks)$ in an infinite series proceeding according to positive integral powers of u.* Let

$$f(ks) = c_0 + c_1 u + c_2 u^2 + \ldots = \Sigma . c_j u^j.$$

Then

$$C_i = \frac{k^{-i}}{2\pi}\Sigma . \int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n} c_j u^j\, du.$$

The definite integral

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n}\, du$$

is a function of n and i: with Cauchy we will denote it by $[n]_i$. Then by taking the derivative of the quantity, under the integral sign, j times with respect to i, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n} u^j\, du = (\sqrt{-1})^j D_i^j . [n]_i .$$

Whence we have the symbolic expression for C_i,

$$C_i = k^{-i} f(k\varepsilon^{-D_i}) . [n]_i .$$

But we have

$$\varepsilon^{D_i} = 1 + \Delta, \quad \varepsilon^{-D_i} = \frac{1}{1+\Delta},$$

Δ here denoting the characteristic of finite differences with respect to the variable i, and not the distance between the two planets. Let

$$\nabla = \frac{\Delta}{1+\Delta}, \text{ then } \varepsilon^{-D_i} = 1 - \nabla.$$

Making these substitutions, we have

$$C_i = k^{-i} f(k - k\nabla) . [n]_i .$$

By successive integrations by parts, making the integration always bear on the first factor, we find the value of the definite integral,

$$\frac{1}{2\pi}\int_0^{2\pi} \varepsilon^{-iu\sqrt{-1}}(1-\varepsilon^{u\sqrt{-1}})^{-n}\, du = [n]_i = \frac{n(n+1)\ldots(n+i-1)}{1.2\ldots i}.$$

* This is the assumption which leads to the semi-convergent series representing the value of C_i. Its allowableness is shown by the fact of the relative smallness of the definite integral which ought to be added to complete the truncated series, when i is tolerably large and the number of terms taken into account is not too great. As Cauchy has treated this point at length, in his memoir first mentioned above, I have thought it unnecessary to say more about it here.

When the function $f(k - k\nabla)$ is developed in ascending powers of ∇, the general term of C_i will be proportional to

$$\nabla^j.[n]_i = \frac{\Delta^j}{(1+\Delta)^j}.[n]_i = \Delta^j.[n]_{i-j} = [n-j]_i.$$

And, developing the last expression for C_i, and employing accents, attached to f, to denote differentiation of the form of f, we have

$$C_i = k^{-i}\Big\{ f(k)[n]_i - kf'(k)[n-1]_i + \frac{1}{1.2}k^2 f''(k)[n-2]_i - \frac{1}{1.2.3}k^3 f'''(k)[n-3]_i + \ldots \Big\}.$$

This may also be written

$$C_i = k^{-i}[n]_i \Big\{ f(k) - f'(k).k\frac{n-1}{i+n-1} + \frac{1}{1.2}f''(k).k^2\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)} - \frac{1}{1.2.3}f'''(k).k^3\frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)} + \ldots \Big\}.$$

We may employ the Γ function to express $[n]_i$, and then

$$[n]_i = \frac{\Gamma(i+n)}{\Gamma(n)\Gamma(i+1)}.$$

In practice, n will have some one of the following series of values,

$$\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2},\ \text{etc.};$$

and it is well known that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},\ \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi},\ \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi},\ \text{etc.}$$

When i is a tolerably large integer, we may use the semi-convergent series

$$\log\Gamma(i+n) = \frac{1}{2}\log(2\pi) + \left(i+n-\frac{1}{2}\right)\log(i+n-1)$$
$$+ M\Big\{ -(i+n-1) + \frac{B_1}{1.2}\frac{1}{i+n-1} - \frac{B_3}{3.4}\frac{1}{(i+n-1)^3} + \frac{B_5}{5.6}\frac{1}{(i+n-1)^5} - \ldots \Big\},$$
$$\log\Gamma(i+1) = \frac{1}{2}\log(2\pi) + \left(i+\frac{1}{2}\right)\log i$$
$$+ M\Big\{ -i + \frac{B_1}{1.2}\frac{1}{i} - \frac{B_3}{3.4}\frac{1}{i^3} + \frac{B_5}{5.6}\frac{1}{i^5} - \ldots \Big\},$$

where M is the modulus of common logarithms, and B_1, B_3, etc., are the numbers of Bernoulli. Thence is derived

$$\log\frac{\Gamma(i+n)}{\Gamma(i+1)}=\left(i+\frac{1}{2}\right)\log\frac{i+n-1}{i}+(n-1)\log(i+n-1)$$
$$-M\left\{n-1+\frac{B_1}{1.2}\left[\frac{1}{i}-\frac{1}{i+n-1}\right]-\frac{B_3}{3.4}\left[\frac{1}{i^3}-\frac{1}{(i+n-1)^3}\right]\right.$$
$$\left.+\frac{B_5}{5.6}\left[\frac{1}{i^5}-\frac{1}{(i+n-1)^5}\right]-\ldots\right\}$$
$$=\left(i+\frac{1}{2}\right)\log\frac{i+n-1}{i}+(n-1)\log(i+n-1)$$
$$-M(n-1)\left\{1+\frac{1}{12}\,\frac{1}{i(i+n-1)}-\frac{1}{360}\,\frac{i^2+i(i+n-1)+(i+n-1)^2}{i^3(i+n-1)^3}\right.$$
$$+\frac{1}{1260}\,\frac{i^4+i^3(i+n-1)+i^2(i+n-1)^2+i(i+n-1)^3+(i+n-1)^4}{i^5(i+n-1)^5}$$
$$\left.-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right\}.$$

The first term of the last expression for C_i affords a first approximation to its value, correct, so to speak, to quantities of the order of $\frac{1}{i}\cdot$ Then

$$C_i=k^{-i}[n]_i f(k).$$

In like manner, the two terms at the beginning afford an approximation correct to quantities of the order of $\frac{1}{i^2}\cdot$ Here we can effect a remarkable reduction; for, on comparing the two terms in question with the two first terms of Taylor's theorem, we see that, to the same degree of approximation, we may write

$$C_i=k^{-i}[n]_i f\left(\frac{i}{i+n-1}k\right).$$

No more labor is involved in employing this expression than in the preceding.

IV.

In this condition Cauchy leaves the subject, but we may go a step farther. In the cases which come up in practice $f(k)$ is always such a function that successive differentiation immensely complicates it; so that it is scarcely possible to go beyond $f''(k)$. Hence a great deal of labor is saved, if, instead of attempting to calculate $f'(k), f''(k)$, etc., we substitute the calculation of $f(k)$ for several values of the argument k. It is easy to perceive that, in general, all the derivatives $f'(k), f''(k)$, etc., may be eliminated from the expression for C_i. For, cutting the series off at the term which contains $f^{(2p)}(k)$ as a factor, we may suppose that, to the same degree of approximation,

$$C_i=k^{-i}[n]_i\{x_0 f(k-ky_0)+x_1 f(k-ky_1)+\ldots+x_p f(k-ky_p)\},$$

where $x_0, x_1, \ldots x_p$ and $y_0, y_1, \ldots y_p$ are unknowns to be suitably determined.

By developing this expression for C_i in powers of k and comparing it with the previous expression, we get the following system of simultaneous equations for determining the unknowns $x_0, x_1, \ldots x_p, y_0, y_1, \ldots y_p$:

$$x_0 + x_1 + x_2 + \ldots + x_p = 1,$$
$$x_0 y_0 + x_1 y_1 + x_2 y_2 + \ldots + x_p y_p = \frac{(n-1)}{i+n-1},$$
$$x_0 y_0^2 + x_1 y_1^2 + x_2 y_2^2 + \ldots + x_p y_p^2 = \frac{(n-1)(n-2)}{(i+n-1)(i+n-2)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_0 y_0^{2p+1} + x_1 y_1^{2p+1} + x_2 y_2^{2p+1} + \ldots + x_p y_p^{2p+1} = \frac{(n-1)\ldots(n-2p-1)}{(i+n-1)\ldots(i+n-2p-1)}.$$

For the sake of brevity we will denote the right-hand members of these equations as a_0, a_1, $a_2 \ldots a_{2p+1}$. The solution of these equations is very elegant. According to the theorem of Bezout, the degree of the final equation, obtained by elimination, would be $=(2p+2)!$ But, as we shall see, the solution depends on that of an equation of the $(p+1)^{\text{th}}$ degree, whose roots are the values of the several unknowns $y_0, y_1 \ldots y_p$; and there is practically but one solution.

Let us suppose that the values of the y's, in any particular solution, are the roots of the equation

$$y^{p+1} + s_1 y^p + s_2 y^{p-1} + \ldots + s_{p+1} = 0,$$

so that

$$s_1 = -(y_0 + y_1 + y_2 + \ldots + y_p),$$
$$s_2 = y_0 y_1 + y_0 y_2 + y_1 y_2 + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$s_{p+1} = (-1)^{p+1} y_0 y_1 y_2 \ldots y_p.$$

Hence, y_q denoting any one of the y's, we must have

$$y_q^{p+1} + s_1 y_q^p + s_2 y_q^{p-1} + \ldots + s_{p+1} = 0.$$

Now, in the group of equations to be solved, multiply the equation, whose second member is a_{p+1}, by 1, the one, whose second member is a_p, by s_1, and so on until the first equation is multiplied by s_{p+1}. Then, by adding all the equations thus obtained, the first member of the resulting equation vanishes, and we have

$$a_{p+1} + a_p s_1 + a_{p-1} s_2 + \ldots + a_0 s_{p+1} = 0.$$

By cutting off the first equation and adding to the group the equation whose second member is a_{p+2}, and writing x_0 for $x_0 y_0$, x_1 for $x_1 y_1$, and so on, we obtain a group which differs from the former only in the second members. Hence we have, from this group, the equation

$$a_{p+2} + a_{p+1} s_1 + a_p s_2 + \ldots + a_1 s_{p+1} = 0.$$

And, in a similar manner,

$$a_{p+3} + a_{p+2} s_1 + a_{p+1} s_2 + \ldots + a_2 s_{p+1} = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{2p+1} + a_{2p} s_1 + a_{2p-1} s_2 + \ldots + a_p s_{p+1} = 0.$$

These $p+1$ linear equations suffice to determine the values of $s_1, s_2 \ldots s_{p+1}$, the coefficients of the equation of the $(p+1)^{\text{th}}$ degree, which has, as its roots, the values of the unknowns y. These values, being obtained and substituted in the first $(p+1)$ equations of the original group, we have a group of $(p+1)$ linear equations for determining the $(p+1)$ unknowns $x_0, x_1 \ldots x_p$. It is plain that all possible solutions of the group of equations are obtained by permuting between themselves the roots of the equation which gives the values of the y's; and, as thus, to each root, corresponds its special value of the x's, and the order in which the several terms of C_i stand, is of no import, it is clear that, practically at least, but one solution exists.

In practice, p never need exceed 2. For $p=0$, the solution has already been given. For $p=1$, we have

$$\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}+\frac{n-1}{i+n-1}s_1+s_2=0,$$

$$\frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)}+\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}s_1+\frac{n-1}{i+n-1}s_2=0.$$

The solution of these gives

$$s_1=-2\frac{n-2}{i+n-3}, \quad s_2=\frac{(n-1)(n-2)}{(i+n-2)(i+n-3)}.$$

Thus the equation which contains the values of the y's is

$$y^2-2\frac{n-2}{i+n-3}y+\frac{(n-1)(n-2)}{(i+n-2)(i+n-3)}=0.$$

Whence the two values of y are

$$y=\frac{n-2\pm\sqrt{\frac{(2-n)(i-1)}{i+n-2}}}{i+n-3};$$

and the corresponding values of x are

$$x=\frac{1}{2}\left[1\pm\frac{i-n+1}{i+n-1}\sqrt{\frac{i+n-2}{(2-n)(i-1)}}\right].$$

In many cases these values will be imaginary, which, however, does not hinder their use, as k is imaginary.

For $p=2$, we have

$$\frac{(n-1)(n-2)(n-3)}{(i+n-1)(i+n-2)(i+n-3)}+\frac{(n-1)(n-2)}{(i+n-1)(i+n-2)}s_1+\frac{n-1}{i+n-1}s_2+s_3=0,$$

$$\frac{(n-2)(n-3)(n-4)}{(i+n-2)(i+n-3)(i+n-4)}+\frac{(n-2)(n-3)}{(i+n-2)(i+n-3)}s_1+\frac{n-2}{i+n-2}s_2+s_3=0,$$

$$\frac{(n-3)(n-4)(n-5)}{(i+n-3)(i+n-4)(i+n-5)}+\frac{(n-3)(n-4)}{(i+n-3)(i+n-4)}s_1+\frac{n-3}{i+n-3}s_2+s_3=0.$$

The solution of these equations gives

$$s_1 = -3\frac{n-3}{i+n-5}, \quad s_2 = 3\frac{(n-2)(n-3)}{(i+n-4)(i+n-5)}, \quad s_3 = -\frac{(n-1)(n-2)(n-3)}{(i+n-3)(i+n-4)(i+n-5)}.$$

The equation, which has, for its roots, the values of the y's, is

$$y^3 - 3\frac{n-3}{i+n-5}y^2 + 3\frac{(n-2)(n-3)}{(i+n-4)(i+n-5)}y - \frac{(n-1)(n-2)(n-3)}{(i+n-3)(i+n-4)(i+n-5)} = 0.$$

By comparing this with the equation for the case where $p = 1$, we readily see what the equation would be for higher values of p.

As an example, suppose it were required to find the coefficient of z^{18} in the expansion of

$$[1 - 2\mathfrak{a}\cos(u - Q) + \mathfrak{a}^2]^{-\frac{3}{2}}.$$

Here the form of $f(s)$ is

$$f(s) = \left(1 - \frac{\mathfrak{a}^2 k}{s}\right)^{-\frac{3}{2}}\left[1 - \frac{e}{2}\left(s + \frac{1}{s}\right)\right]\varepsilon^{9e\left(s-\frac{1}{s}\right)}.$$

In the first place let two terms in the final expression for C_i be regarded as sufficient, that is, put $p = 1$. Then $i = 18$, $n = \frac{3}{2}$, and the two values of y are

$$y = \frac{-1 \pm 2\sqrt{\frac{17}{35}}}{33};$$

and the corresponding value of x is

$$x = \frac{1}{2}\left(1 \pm \frac{35}{37}\sqrt{\frac{35}{17}}\right).$$

Thus the expression for C_i is

$$C_{18} = k^{-18}\left[\frac{3}{2}\right]_{18}\{1.17865 f(0.9880647k) - 0.17865 f(1.0725413k)\}.$$

The error of this is of the order of $\frac{1}{i^4}$, while, in case $p = 0$, which gives the formula

$$C_{18} = k^{-18}\left[\frac{3}{2}\right]_{18} f\left(\frac{36}{37}k\right),$$

which Cauchy employed, the error is of the order of $\frac{1}{i^2}$.

In case we make $p = 2$, and thus have three terms in the formula for C_i, the roots of the cubic

$$y^3 + \frac{9}{29}y^2 + \frac{9}{31.29}y + \frac{9}{33.31.29} = 0$$

must be found. They are

$$y_0 = +0.00804343, \quad y_1 = -0.04617994, \quad y_2 = -0.27220828.$$

The linear equations for determining the x's are

$$\begin{aligned} x_0 + \quad x_1 + \quad x_2 &= 1, \\ 0.0804343x_0 - 0.4617994x_1 - 2.722083x_2 &= 0.2702703, \\ 0.0064697x_0 + 0.2132586x_1 + 7.409736x_2 &= -0.0772201. \end{aligned}$$

The solution of which gives

$$x_0 = +1.3426685, \quad x_1 = -0.3408857, \quad x_2 = -0.0017828.$$

Thus, in this case, we should have

$$C_{18} = k^{-18}\left[\frac{3}{2}\right]_{18}\{1.3426685f(0.9919566k) - 0.3408857f(1.04617994k) - 0.0017828f(1.2722083k)\}.$$

The error of this formula is only of the order of $\frac{1}{i^6}$.

In further illustration of this method, let us find the value $\mathfrak{b}_{\frac{3}{2}}^{(18)}$ of the coefficient of cos 18θ in the periodic development of

$$(1 - 2\alpha\cos\theta + \alpha^2)^{-\frac{3}{2}},$$

where $\alpha = 0.723332$ the ratio of the mean distances of Venus and the earth from the sun. Here the form of $f(s)$ is simply

$$f(s) = \left(1 - \frac{\alpha}{s}\right)^{-\frac{3}{2}}$$

Let us take the formula where $p = 1$. We have

$$\mathfrak{b}_{\frac{3}{2}}^{(18)} = 2C_{18} = 2\left[\frac{3}{2}\right]_{18}\alpha^{18}\left\{1.17865\left(1 - \frac{\alpha^2}{0.9880647}\right)^{-\frac{3}{2}} - 0.17865\left(1 - \frac{\alpha^2}{1.0725413}\right)^{-\frac{3}{2}}\right\}.$$

The value of $\left[\frac{3}{2}\right]_{18}$ will be found in the table at the end of this memoir. And, on the substitution of the numerical values, we get $\mathfrak{b}_{\frac{3}{2}}^{(18)} = 0.090880$. Delaunay, in his memoir,* has 0.090876.

In the case where the function to be developed contains the anomalies of two planets, after the value of C_i has been obtained corresponding to j points evenly distributed on the circumference with reference to the variable l' or the variable u', the value of $C_{i,i'}$ results by employing the method of mechanical quadratures: the formula in the first case being

$$C_{i,i'} = \frac{1}{j}\Sigma \,.\, C_i z'^{-i'},$$

and, in the second,

$$C_{i,i'} = \frac{1}{j}\Sigma \,.\, C_i \frac{r'}{a'} s'^{-i'}.$$

In the annexed table are given the common logarithms of the function $[n]_i$, for n as far as $n = \frac{9}{2}$, and for i, as far as $i = 30$. As they have been computed with the ten-figure logarithms of Vega's *Thesaurus Logarithmorum*, it is to be presumed that they are correct, in nearly every case, to half a unit in the last place.

Connaissance des Temps, 1862.

TABLE OF THE VALUES OF LOG $[n]_i$.

i.	$n=\frac{1}{2}$.	$n=\frac{3}{2}$.	$n=\frac{5}{2}$.	$n=\frac{7}{2}$.	$n=\frac{9}{2}$.
1	9.6989700	0.1760913	0.3979400	0.5440680	0.6532125
2	9.5740313	0.2730013	0.6409781	0.8962506	1.0925452
3	9.4948500	0.3399481	0.8170693	1.1594920	1.4283373
4	9.4368581	0.3911006	0.9553720	1.3703454	1.7013386
5	9.3911006	0.4324933	1.0693154	1.5464366	1.9317875
6	9.3533120	0.4672554	1.1662254	1.6977043	2.1313599
7	9.3211273	0.4972186	1.2505463	1.8303299	2.3074511
8	9.2930986	0.5235475	1.3251799	1.9484292	2.4650590
9	9.2682750	0.5470286	1.3921267	2.0548845	2.6077265
10	9.2459986	0.5682179	1.4528245	2.1517945	2.7380602
11	9.2257953	0.5875231	1.5083418	2.2407356	2.8580356
12	9.2073118	0.6052519	1.5594944	2.3229224	2.9691860
13	9.1902785	0.6216423	1.6069190	2.3993107	3.0727266
14	9.1744842	0.6368822	1.6511227	2.4706666	3.1696366
15	9.1597610	0.6511227	1.6925154	2.5376134	3.2607171
16	9.1459727	0.6644866	1.7314334	2.6006651	3.3466317
17	9.1330077	0.6770758	1.7681562	2.6602508	3.4279367
18	9.1207733	0.6889750	1.8029183	2.7167322	3.5051026
19	9.1091914	0.7002560	1.8359186	2.7704171	3.5785315
20	9.0981960	0.7109799	1.8673271	2.8215696	3.6485694
21	9.0877306	0.7211990	1.8972903	2.8704181	3.7155162
22	9.0777464	0.7309589	1.9259355	2.9171615	3.7796337
23	9.0682010	0.7402989	1.9533737	2.9619739	3.8411517
24	9.0590577	0.7492537	1.9797027	3.0050085	3.9002732
25	9.0502837	0.7578539	2.0050085	3.0464012	3.9571780
26	9.0418506	0.7661264	2.0293679	3.0862727	4.0120267
27	9.0337327	0.7740954	2.0528490	3.1247310	4.0649628
28	9.0259073	0.7817822	2.0755129	3.1618728	4.1161153
29	9.0183542	0.7892062	2.0974148	3.1977853	4.1656007
30	9.0110560	0.7963858	2.1186051	3.2325484	4.2135252

Seminvariants and Symmetric Functions.

By Capt. P. A. MacMahon, R. A.

In this paper I consider the binary quantic with derived coefficients
$$a_0x^n + na_1x^{n-1}y + n(n-1)a_2x^{n-2}y^2 + n(n-1)(n-2)a_3x^{n-3}y^3 + \ldots$$
or putting y equal to unity, it may be written symbolically thus, viz.
$$(a_0, a_1, a_2 \ldots \between \frac{x^n}{1-\partial_x})$$
where ∂_x stands for $\frac{d}{dx}$.

Where it is convenient a_0 will be supposed unity, and sometimes the literal coefficients will be taken to be $a, b, c, d, \ldots$ An advantage of discussing the equation in this form is that the seminvariant operator
$$\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots$$
belonging to the quantic with binomial coefficients $(1, a_1, a_2, \ldots \between x, 1)^n$ becomes in the present case $\qquad \partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots$
an operator which, when considered in connection with the equation
$$(1, a_1, a_2, \ldots \between x, 1)^n = 0$$
is such that it reduces all the non-unitary-partition symmetric functions to zero, and further reduces a symmetric function whose partition contains r units to a similar one containing $r-1$ units; consequently the solutions of the partial differential equation $\qquad \partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots = 0$
are contained in the tables of symmetric functions which have been already calculated as far as the twelfth degree and amply verified.

We thus arrive at the result that every seminvariant of the equation with derived coefficients is a non-unitary-partition symmetric function of the roots of the equation $(1, a_1, a_2, \ldots \between x, 1)^n = 0$. This paper is based upon the analogy shown to exist between seminvariants and symmetric functions, and it will

appear that it gives us a universal method for forming an asyzygetic series of seminvariants of a given weight in a form which discovers the syzygants where such exist.

In lieu of considering the correspondence between the two quantics

$$(a_0, a_1, \ldots \between \frac{x^n}{1-\partial_x}), \quad (a_0, a_1 \ldots \between x, 1)^n$$

we might have discussed in conjunction the quantics with binomial and exponential coefficients, viz. $(a_0, \ldots \between x, 1)^n$ and $(a_0, a_1 \frac{1}{2!}a_2, \frac{1}{3!}a_3, \ldots \between x, 1)^n$, since the non-unitary-partition symmetric functions of the equation formed by equating the latter quantic to zero are seminvariants of the former; but since the converse is not true unless the seminvariants be affected with numerical factors, the method here adopted seemed the better one. The notation is settled so as to be as nearly as possible in accordance with that of Mr. Hammond (*Amer. Jour.*, V, No. 3); viz. $q_2, c_3, q_4, c_5, q_6, \ldots$ are taken to represent the protomorphs or primary groundforms, the suffix denoting at once the extent (*i. e.* rank or weight of highest letter) and the weight of the source.

In Mr. Hammond's paper these are respectively $H, C_3, Q_4, C_5, Q_6, \ldots$ and refer to the quantic with binomial coefficients; any protomorph expression of a seminvariant of the quantic with derived coefficients which involves c's and q's may be transformed so as to belong to the quantic $(a_0, a_1, \ldots \between x, 1)^n$ by means of the equations

$$C_{2m+1} = (2m)!\,c_{2m+1}$$
$$Q_{2m} = m\,(2m-1)!\,q_{2m}$$

the simply numerical factor being rejected; so that the present results may be readily identified with those of Mr. Hammond. We find

$$\begin{aligned}
q_2 &= 2a_2 - a_1^2, \\
c_3 &= 3a_3 - 3a_1a_2 + a_1^3, \\
q_4 &= 2a_4 - 2a_1a_3 + a_2^2, \\
c_5 &= 5a_5 - 5a_1a_4 + a_2a_3 + 2a_1^2a_3 - a_1a_2^2, \\
q_6 &= 2a_6 - 2a_1a_5 + 2a_2a_4 - a_3^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
q_{2m} &= 2a_{2m} - 2a_1a_{2m-1} + 2a_2a_{2m-2} - \ldots + (-)^{\frac{1}{2}n}a_m^2, \\
c_{2m+1} &= (2m+1)a_{2m+1} - (2m+1)a_1a_{2m} + (2m-3)a_2a_{2m-1} \\
&\quad - (2m-5)a_3a_{2m-2} + \ldots + a_1\{2a_1a_{2m-1} - 2a_2a_{2m-2} \\
&\quad + 2a_3a_{2m-3} - \ldots - (-)^{\frac{1}{2}n}a_m^2\}
\end{aligned}$$

expressions in which the numbers are much lower than in the case of the binomial quantic; representing the symmetric functions of $(1, a_1, \ldots \between x, 1)^n = 0$ by their partitions in ( ), we have

$$\begin{array}{ll} q_2 = -(2), & q_6 = -(2^3) \\ c_3 = -(3), & c_7 = -(32^2) \\ q_4 = +(2^2), & q_8 = +(2^4) \\ c_5 = +(32), & c_9 = +(32^3) \\ q_{2m} = (-)^m(2^m); & c_{2m+1} = (-)^m(32^{m-1}); \end{array}$$

since the partition involved for any weight is the simplest non-unitary one of the number representing the weight, we obtain conclusive evidence that the most advantageous baseforms are those of alternately two and three degrees. For weight w we have a seminvariant of extent = weight corresponding to each non-unitary partition of w, and all these are linearly independent; the degree of each is given by the highest part in the corresponding partition, or, what comes to the same thing, by the number of parts in the conjugate partition; but such a system of linearly independent seminvariants of a given weight is not the simplest that can be formed, for by adding to any seminvariant certain numerical multiples of the seminvariants of lower degrees a certain number of the terms of highest extent can be made to vanish, and we can choose the multipliers so that a maximum number shall vanish; we thus obtain another simple system of the same number of forms, and each form of a certain degree in one system will produce another of the same degree in the other, viz. there will be a one to one correspondence between the two systems. Hence arises a theorem for the number of linearly independent seminvariants of a given degree and weight, considered '*inter alia*'; thus for degree-weight (j, w) the number is equal to the number of non-unitary partitions of w which contain a part j and no part greater than j; that is, it is the coefficient of x^w in $\frac{x^j}{(1-x^2)(1-x^3)\ldots(1-x^j)}$, which therefore is the generating function for the number of seminvariants in question.

In the case $(j, w) = (5, 11)$ the expanded form is

$$x^5 + x^7 + x^8 + 2x^9 + 2x^{10} + 3x^{11} + \ldots$$

showing that the number is three, which we know to be the case, as they must correspond to the partitions $542.53^2.52^3$. But considering the asyzygetic seminvariants of degree-weight (5.11) '*ab alia*,' we know from Professor Cayley's rule that there are four; and since there are but three when all those of the

weight 11 are considered '*inter alia,*' it follows that there must be a syzygant of degree-weight (5.11); using [ ] to denote degree-weight, we know from Mr. Hammond's late researches that

$$[2.4][3.7]-[2.6][3.5]=[1.0][4.11].$$

Syzygants of any degree-weight can therefore be detected in this manner.

Mr. Durfee (Johns Hopkins University Circulars, Dec. 1882) has shown the most advantageous way of arranging the terms of a symmetric function, viz. representing the combinations of coefficients by their partitions, the first half (not counting the self-conjugate ones) in ascending order as to number of parts, and in dictionary order in each set of so many parts; then the self-conjugate partitions; then the conjugates of the first half in inverse order.

It seems natural therefore to arrange the terms of the single-partition seminvariants according to the same plan; the ending terms then will have partitions which are the conjugates of the partitions of the symmetric functions to which they respectively correspond, and this will remain so after the asyzygetic series has been formed; consequently an ending term must have a partition, the conjugate of a non-unitary partition.

The literal expressions of the asyzygetic seminvariants up to weight 12 are annexed, the first set having reference to the quantic with derived coefficients, and the second set referring to the quantic $(a_0, a_1 \ldots \between x, y)^n$; the terms have been arranged as explained above, with the exception of the case weight 11, and a very singular symmetry becomes evident, viz. the tables are of exactly the same shape when inverted; this shows that to each seminvariant corresponds another, the partitions of whose terms are the conjugates of its own; it follows therefore that whenever there is an odd number of asyzygetic seminvariants, there must be at least one self-conjugate seminvariant; that is, one containing no term without at the same time containing another whose partition is conjugate to its own; such forms are shown by the dark vertical lines; the pairs of conjugate seminvariants are arranged symmetrically about these in the same way exactly as the pairs of terms are arranged about the self-conjugate terms; the asterisks are placed in those squares which are empty in virtue either (1) the ending term denoting the degree of the form, or (2) the leading term denoting the extent. The asterisks and blank spaces each enjoy the same symmetry.

A slightly different arrangement was found preferable in the case of weight 11; the first half of the terms are altered so as to bring those of like extent

together. The leading terms are the non-unitary partitions, and the law of their correspondence with the ending terms appears to be difficult. With one exception the following law holds, viz. the m^{th} leading term from the left has for its ending term the conjugate of the m^{th} leading term from the right; the exception is in the case of the self-conjugate seminvariants of weight 11, viz. an interchange of ending terms occurs between them; if this were not so they could not both be self-conjugate.

Proceeding to develop the protomorph expressions of these seminvariants as far as weight 11, the following scheme shows for each weight: (i) the protomorph expressions of the single partition forms, (ii) the partition expressions of the asyzygetic series, (iii) the protomorph expression of the latter; the forms are numbered successively i, ii, iii . . .

WEIGHT 2.

(i) (ii) and (iii)

	q_2
No. 1. (2)	-1

WEIGHT 3.

(i) (ii) and (iii)

	c_3
No. 1. (3)	-1

WEIGHT 4.

(i)

	q_4	q_2^2
(4)	-2	$+1$
(2^2)	$+1$	

(ii)

	(2^2)	(4)
No. 1	$+1$	
No. 2	$+2$	$+1$

(iii)

	q_2^2	q_4
No. 1		$+1$
No. 2	$+1$	

WEIGHT 5.

(i)

	c_5	c_3q_2
(5)	-1	$+1$
(32)	$+1$	

(ii)

	(32)	(5)
No. 1	$+1$	
No. 2	$+1$	$+1$

(iii)

	c_3q_2	c_5
No. 1		$+1$
No. 2	$+1$	

WEIGHT 6.

	q_6	q_4q_2	c_3^2	q_2^3
(6)	−3	+3		−1
(42)	+3	−1		
2(3²)	+3	−3	+1	+1
(2³)	−1			

	(2³)	(3²)	(42)	(6)
No. 1	−1			
No. 2	−3	−2		
No. 3	+6	−2	+3	
No. 4	+6		+3	+1

	q_2^3	c_3^2	q_2q_4	q_6
No. 1				+1
No. 2	−1	−1	+3	
No. 3	+1	+1		
No. 4	+1			

Here, for the first time, a syzygant is evident, for No. 2 weight 6 is equal to

$$-3(2^3)-2(3^2)=-q_2^3-c_3^2+3q_2q_4,$$

the left-hand side of this equation is obviously of degree 3, whereas the right-hand side is of degree 6; consequently it must divide out by a_0^3, leaving a syzygant of degree 3; this is, "*mutatis mutandis*," T in Mr. Hammond's notation, a ground-source for the quartic and all higher quantics; it is convenient to write it T_{364} so that the degree, weight and rank will be denoted.

No. 3 weight 6 is $+6(2^3)-2(3^2)+3(42)=q_2^3+c_3^2$,
showing that the right-hand side must divide out by a_0^2 leaving a ground syzygant for the cubic, which, "*mutatis* mutandis*," is Δ_{463}.

But since it may also be written

$$-T_{364}+3q_2q_4,$$

we see that Δ_{463} is not a ground-source for the quartic or any higher quantic.

WEIGHT 7.

(i)

	c_7	c_5q_2	q_4c_3	$c_3q_2^2$
(7)	−1	+1	+1	−1
(52)	+1		−1	
(43)	+1	−1	+1	
(32²)	−1			

(ii)

	(32²)	(43)	(52)	(7)
No. 1	−1			
No. 2	−1	−1		
No. 3	−1		−1	
No. 4	−2	−1	−2	−1

(iii)

	$c_3q_2^2$	q_4c_3	c_5q_2	c_7
No. 1				+1
No. 2		−1	+1	
No. 3		+1		
No. 4	+1			

No. 2 weight 7 is $-(32^2)-(43)=-q_4c_3+c_5q_2,$
showing that a_0 must be a factor, leaving P_{475} a ground-source for the quintic and all higher quantics.

* This will be understood in future.

WEIGHT 8.

(i)

	q_8	q_6q_2	c_5c_3	q_4^2	$q_4q_2^2$	$c_3^2q_2$	q_2^4
(8)	−4	+4		+2	−4		+1
(62)	+4	−1		−2	+1		
(53)	+4	−4	+1	−2	+4	−1	−1
(4^2)	+2	−2		+1			
(42^2)	−4	+1					
$2(3^22)$	−8	+5	−2	+4	−5	+1	+1
(2^4)	+1						

(ii)

	(2^4)	(3^22)	(42^2)	(4^2)	(53)	(62)	(8)
No. 1	+1						
No. 2	+8	+2					
No. 3	+12	−2	+5				
No. 4	+6		+2	+1			
No. 5	+12	+2	+3		+2		
No. 6	−12	−2			−2	+3	
No. 7	+48		+20	+10		+4	+1

(iii)

	q_2^4	$c_3^2q_2$	$q_4q_2^2$	q_4^2	c_5c_3	q_6q_2	q_8
No. 1							+1
No. 2	+1	+1	−5	+4	−2	+5	
No. 3	−1	−1	+5	−4	+2		
No. 4				+1			
No. 5	−1	−1	+3				
No. 6	+1	+1					
No. 7	+1						

No. 2 weight 8 has evidently a factor a_0^5 and is a syzygant, say K_{386}, a ground-source for the sextic and all higher quantics.

No. 3 weight 8 has a factor a_0^4 and is a ground-source, say L_{485}, deg. ord. (4.4) for the quintic, but since it may be expressed in the form $-K_{386}+5q_6q_2$ it is not a ground-source for the sextic and higher quantics.

Nos. 5 and 6 reduce respectively to q_2T_{361} and $q_2\Delta_{463}$. The analysis of weight 8 therefore is complete.

WEIGHT 9.

(i)

	c_9	c_7q_2	q_6c_3	c_5q_4	$c_5q_2^2$	$q_4c_3q_2$	c_3^3	$c_3q_2^3$
(9)	−1	+1	+1	+1	−1	−2		+1
(72)	+1		−1	−1		+1		
(63)	+1	−1	+2	−1	+1	−1		
(54)	+1	−1	−1	+1				
(52^2)	−1		+1					
(432)	−2	+1	−1					
(3^3)	−2	+2	−7	+2	−2	+5	−1	−1
(32^3)	+1							

(ii)

	(32^3)	(3^3)	(432)	(52^2)	(54)	(63)	(72)	(9)
No. 1	+1							
No. 2	+2	+6						
No. 3	+2	−6	+2					
No. 4	+3	+1	−2	+5				
No. 5	+3	−1	+4	−1	+2			
No. 6	+3	+6		+3		+2		
No. 7	+6	−6	+3	+6	+3	−2	+3	
No. 8	+6		+3	+6	+3	+1	+3	+1

(iii)

	$c_3q_2^3$	c_3^3	$q_4c_3q_2$	$c_5q_2^2$	c_5q_4	q_6c_3	c_7q_2	c_9
No. 1								+1
No. 2	−1	−1	+5	−2	+2	−7	+2	
No. 3	+1	+1	−5	+2	−2	+5		
No. 4	−1	−1	+5	−2	+2			
No. 5	+1	+1	−5	+2				
No. 6	−1	−1	+3					
No. 7	+1	+1						
No. 8	+1							

No. 2 weight 9 is obviously divisible by a_0^6, and is therefore a syzygant giving R_{397} a ground-source for the septimic and all higher quantics.

No. 3 is, after rejection of the factor a_0^5, S_{496} a ground-source for the sextic and all higher quantics.

No. 5 is a ground-source for the quintic Λ_{595}, a_0^4 being the integrating factor, but since it can be written

$$2c_5 q_4 - 10 q_6 c_3 + 2S_{496},$$

it is not so for the sextic and higher quantics.

No. 4 is now seen to be $2c_5 q_4 - \Lambda_{595}$, which is of the right degree.

No. 6, of degree 6, is $c_3 T_{364}$.

No. 7, of degree 7, is $c_3 \Delta_{463}$.

Weight 10.

(i)

	q_{10}	q_8q_2	c_7c_3	q_6q_4	$q_6q_2^2$	c_5^2	$c_5c_3q_2$	$q_4^2q_2$	$q_4c_3^2$	$q_4q_2^3$	$c_3^2q_2^2$	q_2^5
(10)	−5	+5		+5	−5			−5		+5		−1
(82)	+5	−1		−5	+1			+3		−1		
(73)	+5	−5	+1	−5	+5		−1	+5	−1	−5	+1	+1
(64)	+5	−5		+1	+2			−1				
(62^2)	−5	+1		+2	−1							
$2(5^2)$	+5	−5		−5	+5	+1	−2	+5		−5	+1	+1
(532)	−10	+6	−1	+10	−6	−1	+2	−8	+1	+6	−1	−1
(4^22)	−5	+3		−1								
$2(43^2)$	−10	+10	−2	+4	−7		+2	−4		+5	−1	−1
(42^3)	+5	−1										
$2(3^22^2)$	+15	−7	+2	−12	+7	+1	−2	+8	−1	−6	+1	+1
(2^5)	−1											

(ii)

	(2^5)	$(3^2 2^2)$	(42^3)	(43^2)	$(4^2 2)$	(532)	(5^2)	(62^2)	(64)	(73)	(82)	(10)
No. 1	−1											
No. 2	−15	−2										
No. 3	−20	+2	−7									
No. 4	−20	−2	−3	−2								
No. 5	−60	+2	−21	+2	−8							
No. 6	−40	+4	−8	−2		−2						
No. 7	−30	−2	−9		−3	−2	−2					
No. 8	−60	+4	−27	+2	−10	+2		−5				
No. 9	−60	+2	−27		−12	+2	+2	−6	−3			
No. 10	−60	−2	−21	−1	−6	−4	−2	−3		−2		
No. 11	−120	+2	−60	+1	−30	+4	+2	−18	−9	+2	−3	
No. 12	+120		+60		+30			+20	+10		+5	+1

(iii)

	q_2^5	$c_3^2q_2^2$	$q_4q_2^3$	$q_4c_3^2$	$q_4^2q_2$	$c_5c_3q_2$	c_5^2	$q_6q_2^2$	q_6q_4	c_7c_3	q_8q_2	q_{10}
No. 1												+1
No. 2	−1	−1	+6	+1	−8	+2	−1	−7	+12	−2	+7	
No. 3	+1	+1	−6	−1	+8	−2	+1	+7	−12	+2		
No. 4			+2	+2	−8		−2		+16			
No. 5			−1	−1	+4		+1					
No. 6	+1	+1	−5		+4	−2		+5				
No. 7			−1	−1	+3							
No. 8	−1	−1	+5		−4	+2						
No. 9			+1	+1								
No. 10	−1	−1	+3									
No. 11	+1	+1										
No. 12	+1											

No. 2 weight 10 has a factor a_0^7 and is a ground-source deg. ord. 3.4 for the octavic and all higher quantics; say $S_{3,10,8}$.

No. 3 has a factor a_0^6, leaving a ground-source deg. ord. 4.8 for the septimic, but since it may obviously be written $7q_8q_2 - S_{3,10,8}$ it is not an irreducible form for the octavic and higher quantics.

Nos. 4 and 5 are not really separate forms; the latter is divisible by a_0^4, leaving a ground-source deg. ord. 4.0 for the quintic and all higher quantics.

No. 6 is simply $q_2 K_{386}$.

No. 7 is equal to $q_4 T_{364}$.

No. 8 " " $q_2 L_{485}$.

No. 9 " " $q_4 \Delta_{463}$.

No. 10 " " $q_2^2 T_{364}$.

No. 11 " " $q_2^2 \Delta_{463}$, so that there exist no more ground-sources of weight 10.

Weight 11.

(i)

	c_{11}	c_9q_2	q_8c_3	c_7q_4	$c_7q_2^2$	q_6c_5	$q_6c_3q_2$	$c_5q_4q_2$	$c_5c_3^2$	$c_5q_2^3$	$q_4^2c_3$	$q_4c_3q_2^2$	$c_3^3q_2$	$c_3q_2^4$
(11)	−1	+1	+1	+1	−1	+1	−2	−2		+1	−1	+3		−1
(92)	+1		−1	−1		−1	+1	+1			+1	−1		
(83)	+1	−1	+3	−1	+1	−1	−2	+2		−1	−1	+1		
(74)	+1	−1	−1	+1		−1	+2				−1			
(72^2)	−1		+1			+1	−1							
(65)	+1	−1	−1	−1	+1	+2	−1	−1			+1			
(632)	−2	+1	−2	+2	−1	−1	+1							
(542)	−2	+1	+2			−1								
2(53^2)	−2	+2	−6	+2	−2	−1	+7	−1	−1	+1	+2	−5	+1	+1
(52^3)	+1		−1											
(4^23)	−1	+1	−1	−1		+1								
(432^2)	+3	−1	+1											
6(3^22)	+6	−4	+18	−6	+4	+3	−14	+1	+3	−1	−6	+10	−2	−2
(32^4)	−1													

(ii)

	(32^4)	(3^32)	(432^2)	(4^23)	(52^3)	(53^2)	(542)	(632)	(65)	(72^2)	(74)	(83)	(92)	(11)
No. 1	−1													
No. 2	−6	−6												
No. 3	−6	+6	−4											
No. 4	−2		−1	−1										
No. 5	−4	+6	−10	−6	+14									
No. 6	−4	−6	+2	+2	−6	−2								
No. 7		−6	+3	+2	−3	−4	+1							
No. 8	−8	−6			−8	−2		−2						
No. 9		+6	−6	−3	+6	+4	−3	−1	−1					
No. 10		−6				−6				+2				
No. 11	−6		−2	−1	−6		−2			−2	−1			
No. 12	−12	−6	−3		−12	−6	−3	−2	−2	−3		−2		
No. 13	−24	+6	−12	−6	−24	+6	−12	−1	−1	−12	−6	+2	−3	
No. 14	−24		−12	−6	−24		−12	−4	−4	−12	−6	−1	−4	−1

(iii)

	$c_3q_2^4$	$c_3^3q_2$	$q_4c_3q_2^2$	$q_4^2c_3$	$c_5q_2^3$	$c_5c_3^2$	$c_5q_4q_2$	$q_6c_3q_2$	q_6c_5	$c_7q_2^2$	c_7q_4	q_8c_3	c_9q_2	c_{11}
No. 1														+1
No. 2	+2	+2	−10	+6	+1	−3	−1	+14	−3	−4	+6	−18	+4	
No. 3	−2	−2	+10	−6	−1	+3	+1	−14	+3	+4	−6	+14		
No. 4									−1		+1			
No. 5	−2	−2	+10	−6	−1	+3	+1	−14	−3	+4				
No. 6				+2	−1	−1	+1		+1					
No. 7				+2	−1	−1	+1							
No. 8	+1	+1	−5	+4		−2		+5						
No. 9				−3	+1	+1								
No. 10	+1	+1	−5	−2	+2									
No. 11				+1										
No. 12	−1	−1	+3											
No. 13	+1	+1												
No. 14	+1													

No. 2 clearly has a factor a_0^8, leaving, say $M_{3,11,9}$ a ground-source for the nonic and all higher quantics; this source is deg. ord. $3.3n-22$ for the n^{ic}.

No. 3 is $-a_0^8M_{3,11,9}+4(c_9q_2-q_8c_3)$, and since the expression in ( ) has a factor a_0, it is a ground-source for the octavic, say $N_{4,11,8}$, but not so for the nonic or any higher quantic; it is deg. ord. 4.10 for the octavic.

No. 4 is $c_7q_4-q_6c_5$, the partition expression showing that it must have a factor a_0; thus, No. 4 $=c_7q_4-q_6c_5=a_0F_{4,11,7}$ a ground-source deg. ord. 4.6 for the septimic and all higher quantics; which is Mr. Hammond's result (*vide* Amer. Jour., Vol. V. No. 1).

Nos. 5, 6 and 7 are not separate forms. No. 7 is obviously divisible by a_0^4, and being the simplest of the three, we take as the source of the covariant deg. ord. 5.3 for the quintic, say $\Theta_{5,11,5}$.

Then No. 6 $= \Theta_{5,11,5} + q_6 c_5$.

No. 5 $= 2q_2 R_{397} - 3$(No. 6),

and consequently Nos. 5 and 6 are not ground-sources.

Since, however, we find easily

$$3\Theta = 3q_6 c_5 - 6c_7 q_4 + 14q_8 c_3 + 2q_2 R_{397} - N_{4,11,8}$$

Θ is seen to be expressible in terms of lower groundforms in the case of the octavic; consequently Θ is a ground-source for the sextic and septimic as well as for the quintic, but not for any higher quantic.

No. 8 is equal to $c_3 K_{386}$.
No. 9 " " $3q_4 P_{475} - c_5 T_3$.
No. 10 " " $c_3 L_{485} + 2c_5 \Delta_{463} + 6q_4^2 c_3$.
No. 12 " " $c_3 q_2 T_{364}$.
No. 13 " " $c_3 q_2 \Delta_{463}$,

which completes the consideration of weight 11.

In the calculation of single-partition seminvariants in terms of protomorphs, Mr. Hammond's other operators (*vide* Proceedings London Math. Soc., February, 1883) may be utilized; these are included in the formula $d_\lambda = [\lambda]$; where $[\lambda]$ stands for the sum of the λ^{th} powers of the roots of the equation

$$x^n + D_1 x^{n-1} + D_2 x^{n-2} + D_3 x^{n-3} + \ldots = 0,$$

and the law of operation of D on ( ) is from his paper

$$D_\kappa(\lambda^l.\mu^m.\nu^n\ldots) = 0, \quad D_\lambda(\lambda^l.\mu^m.\nu^n\ldots) = (\lambda^{l-1}.\mu^m.\nu^n\ldots), \quad D_\lambda(\lambda) = 1.$$

To calculate thus seminvariant $(3^2 2)$, put

$$(3^2 2) = \alpha_1(2^4) + \alpha_2(2^3)(2) + \alpha_3(32)(3) + \alpha_4(2^2)^2 + \alpha_5(2^2)(2)^2 + \alpha_6(3)^2(2) + \alpha_7(2)^4.$$

Operating with d_8 or $\ldots D_2^4 - 4D_3^2 D_2 \ldots$, we find at once $\alpha_1 = -4$, and the lower operators quickly give the other coefficients.

When performed on the literal expression,

$$d_\lambda = \partial_{a_\lambda} + a_1 \partial_{a_{\lambda+1}} + a_2 \partial_{a_{\lambda+2}} + \text{etc.},$$

and clearly any of these operators when performed on a seminvariant will yield seminvariants; the operation of d_3 alone frequently enables us to determine the degree in the protomorphs of any given seminvariant; if ϕ be any form and we find $d_3\phi = 0$, it shows us that ϕ consists entirely of even-weight protomorphs, since d_3 would not make a term containing a 'c' vanish; generally if m

operations of d_3 reduce a seminvariant to zero, its expression in protomorphs can contain no term with more than $m - 1$ odd-weight protomorphs, and must contain a term with that number; if m operations reduce the form to a constant, then the term containing most c's contains m of them and no q; operating with d_2 and d_3 will thus disclose the degree in the protomorphs, which is the end in view.

Putting $a_0 = 0$ in any seminvariant we obtain what Professor Sylvester calls its residue, and he has explained its semi-invariantive character; in the present case the residue is a seminvariant, quâ the elements $a_1, a_2, a_3, a_4 \dots$ that is of the quantic

$$a_1 x^n + n a_2 x^{n-1} + n(n-1) a_3 x^{n-2} + \dots$$

since then the operator becomes

$$\partial_{a_2} + a_2 \partial_{a_3} + a_3 \partial_{a_4} + \dots$$

which is of the same character as

$$\partial_{a_1} + a_1 \partial_{a_2} + a_2 \partial_{a_3} + \dots$$

and differs from it merely in each suffix being increased by unity.

Putting further $a_1, a_2, a_3 \dots a_{m-1}$, each equal to zero, we obtain the m^{th} residue which is a seminvariant of the quantic

$$a_m x^n + n a_{m+1} x^{n-1} + n(n-1) a_{m+2} x^{n-2} + \dots$$

It follows that if $\phi(a_0, a_1, a_2 \dots)$ be a seminvariant, $\phi(a_1, a_2, a_3 \dots)$ is a residue, the latter being obtained from the former by increasing each suffix by unity; the protomorph residues are therefore at once written down, viz. they are

$$2a_3 a_1 - a_2^2$$
$$3a_4 a_1^2 - 3a_3 a_2 a_1 + a_2^3$$
$$2a_5 a_1 - 2a_4 a_2 + a_3^2, \text{ etc.}$$

This simple derivation of the residue groundforms is of some advantage in the exhaustive method of groundform deduction initiated by Professor Cayley; considering the residues quâ the operator

$$a_1 \partial_{a_2} + a_2 \partial_{a_3} + a_3 \partial_{a_4} + \dots$$

the residue syzygants will contain some power of a_1 as a factor; supposing the partitions in $\{\quad\}$ to refer to the seminvariants of the quantic

$$a_1 x^n + n a_2 x^{n-1} + n(n-1) a_3 x^{n-2} + \dots$$

and $\mathfrak{R}$ to denote the residue of the seminvariant which follows it, then

$$\mathfrak{R}(2^\nu) = \{2^{\nu-1}\}; \ \mathfrak{R}(32^\nu) = -a_1\{2^\nu\}; \ \mathfrak{R}(2) = a_1^2,$$

and generally
$$\mathfrak{R}(\lambda^l \mu^m \nu^n \dots) = (-1)^\lambda \{\lambda^{l-1} \mu^m \nu^n \dots\},$$
$$\mathfrak{R}(\lambda \mu^m \nu^n \dots) = (-1)^\lambda a_1^{\lambda-\mu} \{\mu^m \nu^n \dots\};$$

consequently, if in the protomorph expression of any seminvariant there be either an odd weight or a weight two (q_2) protomorph in each term, a_1 will be a factor of the residue; generally if i, t denote respectively the number of odd weight and weight two protomorphs in a term, and the least value of $i+2t$ that occurs in every term be ν, then a_1^ν is a factor of the residue, and conversely if a_1^ν is a factor of the residue, $j+2t$ must be at least equal to ν in every term of the protomorph expression.

From the facts above given it will be seen that starting from a seminvariant of degree weight (j, w), we can immediately write down the residue of a seminvariant of degree-weight $(j, w+j)$; if the former is a member of the weight w asyzygetic series, the latter will not necessarily be a member of the weight $w+j$ series, but, more often than not, will be a linear combination of the members of that series; on this principle the ending terms of a weight w series can be derived from those of the several series of inferior weight in regular succession. Starting from the ending terms of zero weight

$$a_0,\ a_0^2,\ a_0^3,\ a_0^4,\ \ldots$$

by a unit increase of suffixes we derive ending terms a_1^2, a_1^3, a_1^4, ... of weights $2, 3, 4\ldots$; the ending terms of weight 2 are therefore a_1^2, $a_0a_1^2$, $a_0^2a_1^2$, ... and from these we derive ending terms a_2^2, $a_1a_2^2$, $a_1^2a_2^2$, ... of weights $4, 5, 6$; from the ending terms of weight 3, a_1^3, $a_0a_1^3$, $a_0^2a_1^3$, ... we derive a_2^3, $a_1a_2^3$, $a_1^2a_2^3$, ... of weights $6, 7, 8\ldots$; thus the ending terms of weight 4 are a_1^4 and a_2^2, and of weight 5 a_1^5 and $a_1a_2^2$, and so forth; we can thus not only derive the ending terms of higher weights, but also, to numerical factors près, the residues of certain seminvariants having those ending terms. Ex. gr.

Suppose we wish to derive the residues of the weight 8 from the seminvariants of lower weight; we know that the ending terms of weights $0, 2, 3, 4, 5, 6, 7$, are respectively

$$(a_0^*)(a_1^2)(a_1^3)(a_1^4,\ a_2^2)(a_1^5,\ a_1a_2^2)(a_1^6,\ a_1^2a_2^2,\ a_2^3,\ a_3^2)(a_1^7,\ a_1^3a_2^2,\ a_1a_2^3,\ a_1a_3^2);$$

whence we derive by a unit increase of suffixes,

From $a_0^8 \ldots a_1^8$
" $a_0^4a_1^2 \ldots a_1^4a_2^2$
" $a_0^2a_1^3 \ldots a_1^2a_2^3$
" $a_1^4 \ldots a_2^4$
" $a_0^2a_2^2 \ldots a_1^2a_3^2$
" $a_1a_2^2 \ldots a_2a_3^2$
" $a_3^2 \ldots a_4^2$

no other derivation being possible; in fact, if (j, w) be the degree-weight of a lower ending term when multiplied by any positive integer including zero power of a_0, then we must have $j + w = 8$;
the above are the only possible solutions of this equation; so that if we multiply a seminvariant of lower weight by such a power of a_0 that, as regards the ending term: $j + w = 8$,
a unit increase of suffixes will give, to a numerical factor près, the residue of a seminvariant of weight 8 and degree j; if we wish to form the residue of a seminvariant of weight w_2 from a form deg-weight (w_1, j_1), we multiply the latter by $a_0^{w_2 - w_1 - j_1}$ where $w_2 - w_1 - j_1$ is not less than zero, and then a unit increase of suffixes gives, to a numerical factor près, the required residue of a seminvariant of degree-weight $(w_2 - w_1, w_2)$; so that the number of forms of degree-weight (j_2, w_2) in the asyzygetic series must be the equal to the number of forms of weight w_2 that it is possible to derive from the weight $(w_2 - j_2)$ asyzygetic series; this is clearly equal to the number of non-unitary partitions of $w_2 - j_2$ which contain no part greater than j_2; this result is obviously identical with that obtained at the beginning of the paper, viz. that the number is equal to the number of non-unitary partitions of w_2 which contain a part j_2 and no part greater than j_2.

Since the operator $a_0\partial_{a_1} + 2a_1\partial_{a_2} + 3a_2\partial_{a_3} + \ldots$

$$= (a_0\partial_{a_1} + a_1\partial_{a_2} + a_2\partial_{a_3} + \ldots) + (a_1\partial_{a_2} + a_2\partial_{a_3} + a_3\partial_{a_4} + \ldots) + (a_2\partial_{a_3} + a_3\partial_{a_4} + a_4\partial_{a_5} + \ldots) + \ldots$$

we see that the seminvariants of the quantic

$$(a_0, a_1, a_2 \ldots \between x, y)^n,$$

can be expressed in terms of those of the quantics whose type is

$$a_m x^n + n a_{m+1} x^{n-1} + n(n-1) a_{m+2} x^{n-2} + \ldots$$

Asyzygetic Seminvariants.

Tables A *for Quantics with Binomial Coefficients; tables* B *for ditto with Derived Coefficients.*

Tables 1A and 1B.

a_0	+1

Table 2A.

a_2	+1
a_1^2	−1

Table 2B.

a_2	+2
a_1^2	−1

Table 3A.

a_3	+1
a_1a_2	−3
a_1^3	+2

Table 3B.

a_3	+3
a_1a_2	−3
a_1^3	+1

Table 4A.

a_4	+1	
a_1a_3	−4	
a_2^2	+3	+1
$a_1^2a_2$		−2
a_1^4		+1

Table 4B.

a_4	+2	
a_1a_3	−2	
a_2^2	+1	+4
$a_1^2a_2$		−4
a_1^4		+1

Table 5A.

a_5	+1	
a_1a_4	−5	
a_2a_3	+2	+1
$a_1^2a_3$	+8	−1
$a_1a_2^2$	−6	−3
$a_1^3a_2$		+5
a_1^5		−2

Table 5B.

a_5	+5	
a_1a_4	−5	
a_2a_3	+1	+6
$a_1^2a_3$	+2	−3
$a_1a_2^2$	−1	−6
$a_1^3a_2$		+5
a_1^5		−1

Table 6A.

a_6	+1			
a_1a_5	−6			
a_2a_4	+15	+1		
a_3^2	−10	−1	+1	
$a_1^2a_4$		−1	*	
$a_1a_2a_3$		+2	−6	
$a_1^3a_3$		*	+4	
a_2^3		−1	+4	+1
$a_1^2a_2^2$			−3	−3
$a_1^4a_2$				+3
a_1^6				−1

Table 6B.

a_6	+2			
a_1a_5	−2			
a_2a_4	+2	+12		
a_3^2	−1	−9	+9	
$a_1^2a_4$		−6	*	
$a_1a_2a_3$		+6	−18	
$a_1^3a_3$		*	+6	
a_2^3		−2	+8	+8
$a_1^2a_2^2$			−3	−12
$a_1^4a_2$				+6
a_1^6				−1

Table 7A.

a_7	+1			
a_1a_6	−7			
a_2a_5	+9	+1		
a_3a_4	−5	−1	+1	
$a_1^2a_5$	+12	−1	*	
$a_1a_2a_4$	−30	−2	−3	
$a_1a_3^2$	+20	+4	−4	
$a_1^3a_4$		+3	+2	
$a_2^2a_3$		−1	+3	+1
$a_1^2a_2a_3$		−6	+12	−2
$a_1^4a_3$		*	−8	+1
$a_1a_2^3$		+3	−9	−3
$a_1^3a_2^2$			+6	+8
$a_1^5a_2$				−7
a_1^7				+2

Table 7B.

a_7	+7			
a_1a_6	−7			
a_2a_5	+3	+10		
a_3a_4	−1	−6	+6	
$a_1^2a_5$	+2	−5	*	
$a_1a_2a_4$	−2	−4	−6	
$a_1a_3^2$	+1	+6	−6	
$a_1^3a_4$		+3	+2	
$a_2^2a_3$		−1	+3	+12
$a_1^2a_2a_3$		−3	+6	−12
$a_1^4a_3$		*	−2	+3
$a_1a_2^3$		+1	−3	−12
$a_1^3a_2^2$			+1	+16
$a_1^5a_2$				−7
a_1^7				+1

TABLE 8A.

a_8	+1						
a_1a_7	−8						
a_2a_6	+28	+1					
a_3a_5	−56	−3	+3				
a_4^2	+35	+2	−2	+1			
$a_1^2a_6$		−1	*	*			
$a_1a_2a_5$		+3	−9	*			
$a_1a_3a_4$		−1	+1	−8			
$a_2^2a_4$		−3	+18	+6	+1		
$a_1^3a_5$		*	+6	*	*		
$a_2a_3^2$		+2	−12				
$a_1^2a_2a_4$			−15		−2		
$a_2a_3^2$					−1	+1	
$a_1^4a_4$			*	*	+1	*	
$a_1^2a_3^2$			+10	+16	+1	−1	
$a_1a_2^2a_3$				−24	+2	−6	
$a_1^3a_2a_3$				*	−2	+10	
$a_1^5a_3$				*	*	−4	
a_2^4				+9	−1	+4	+1
$a_1^2a_2^3$					+1	−7	−4
$a_1^4a_2^2$						+3	+6
$a_1^6a_2$							−4
a_1^8							+1

TABLE 8B.

a_8	+2						
a_1a_7	−2						
a_2a_6	+2	+10					
a_3a_5	−2	−15	+30				
a_4^2	+1	+8	−16	+4			
$a_1^2a_6$		−5	*	*			
$a_1a_2a_5$		+5	−30	*			
$a_1a_3a_4$		−1	+2	−8			
$a_2^2a_4$		−2	+24	+4	+24		
$a_1^3a_5$		*	+10	*	*		
$a_2a_3^2$		+1	−12				
$a_1^2a_2a_4$			−10		−24		
$a_2a_3^2$					−18	+18	
$a_1^4a_4$			*	*	+6	*	
$a_1^2a_3^2$			+5	+4	+9	−9	
$a_1a_2^2a_3$				−4	+12	−36	
$a_1^3a_2a_3$				*	−16	+30	
$a_1^5a_3$				*	*	−6	
a_2^4				+1	−4	+16	+16
$a_1^2a_2^3$					−2	−14	−32
$a_1^4a_2^2$						+3	+24
$a_1^6a_2$							−8
a_1^8							+1

TABLE 9A.

a_9	+1							
a_1a_8	−9							
a_2a_7	+20	+2						
a_3a_6	−28	−7	+1					
a_4a_5	+14	+5	−1	+1				
$a_1^2a_7$	+16	−2	*	*				
$a_1a_2a_6$	−56	+7	−3	*				
$a_1a_3a_5$	+112	+22	−2	−4				
$a_2^2a_5$		−27	+9	−9	+2			
$a_1a_4^2$	−70	−25	+5	−5				
$a_2a_3a_4$		+45	−17	+32	−5	+1		
$a_1^3a_6$		*	+2	*	*	*		
$a_1^2a_2a_5$		*	−6	+24	−4	*		
$a_1^2a_3a_4$		*	+2	−2	+5	−1		
$a_1^4a_5$		*	*		+2	*		
a_3^3		−20	+8	−18	+3	−1	+1	
$a_1^4a_5$			*	−12		*	*	
$a_1a_2^2a_4$			+6	−51	+5	−3	*	
$a_1^3a_2a_4$			*	+30	−5	+5	*	
$a_1^5a_4$			*	*	*	−2	*	
$a_1a_2a_3^2$			−4	+34	−7	+5	−9	
$a_2^3a_3$					+1	−1	+4	+1
$a_1^3a_3^2$				−20	−2	−2	+6	
$a_1^2a_2^2a_3$					+8	−6	+15	−3
$a_1^4a_2a_3$					*	+4	−24	+3
$a_1^6a_3$					*	*	+8	−1
$a_1a_2^4$					−3	+3	−12	−3
$a_1^3a_2^3$						−2	+17	+11
$a_1^5a_2^2$							−6	−15
$a_1^7a_2$								+9
a_1^9								−2

Table 9B.

a_9	+9							
a_1a_8	−9							
a_2a_7	+5	+14						
a_3a_6	−3	−21	+30					
a_4a_5	+1	+10	−20	+20				
$a_1^2a_7$	+2	−7	*	*				
$a_1a_2a_6$	−2	+7	−30	*				
$a_1a_3a_5$	+2	+11	−5	−20				
$a_2^2a_5$		−9	+30	−15	+40			
$a_1a_4^2$	−1	−10	+20	−20				
$a_2a_3a_4$		+9	−34	+64	−60	+36		
$a_1^3a_6$		*	+10	*	*	*		
$a_1^2a_2a_5$		*	−10	+40	−40	*		
$a_1^2a_3a_4$		*	+2	−10	+30	−18		
$a_1^4a_5$		*	*		+10	*		
a_3^3		−3	+12	−27	+27	−27	+27	
$a_1^4a_5$			*	−10		*	*	
$a_1a_2^2a_4$			+4	−34	+20	−36	*	
$a_1^3a_2a_4$			*	+10	−10	+50	*	
$a_1^5a_4$			*	*	*	−6	*	
$a_1a_2a_3^2$			−2	+17	−21	+45	−51	
$a_2^3a_3$					+2	−6	+24	+24
$a_1^3a_3^2$				−5	−3	−9	+27	
$a_1^2a_2^2a_3$					+8	−6	+45	−36
$a_1^4a_2a_3$					*	+2	−36	+18
$a_1^6a_3$					*	*	+6	−3
$a_1a_2^4$					−2	+6	−24	−24
$a_1^3a_2^3$						−2	+17	+44
$a_1^5a_2^2$							−3	−30
$a_1^7a_2$								+9
a_1^9								−1

Table 10A.

a_{10}	+1											
a_1a_9	−10											
a_2a_8	+45	+1										
a_3a_7	−120	−4	+4									
a_4a_6	+210	+8	−8	+16								
a_5^2	−126	−5	+5	−15	+1							
$a_1^2a_8$		−1	*	*	*							
$a_1a_2a_7$		+4	−12	*	*							
$a_1a_3a_6$		−4	+4	−64	*							
$a_2^2a_6$		−4	+32	+48	*	+1						
$a_1a_4a_5$		+2	−2	+54	−10							
$a_2a_3a_5$		+8	−64	−60	+4	−3	+3					
$a_2a_4^2$		−5	+40		+16	+2	−2	+1				
$a_3^2a_4$				+20	−12			−1	+1			
$a_1^3a_7$			+8	*	*	*	*	*	*			
$a_1^2a_2a_6$			−28		*	−2	*	*	*			
$a_1^2a_3a_5$			+56	+144	+16	+3	−3	*	*			
$a_1a_2^2a_5$				−108	−12	+3	−9	*	*			
$a_1^2a_4^2$			−35	−135	+9	−2	+2	−1				
$a_1^4a_6$				*	*	+1	*	*	*			
$a_1^3a_2a_5$				*	*	−3		*	*			
$a_1a_2a_3a_4$				+180	−76	−1	+1	−2	−6			
$a_1^3a_2a_5$				*	*		+15	*	*			
$a_1^5a_5$				*	*	*	−6	*	*			
$a_2^3a_4$					+48	−3	+18	+2	+4	+1		
$a_1^3a_3a_4$				*	*	+1	−1	+4	+4			
$a_1^2a_2^2a_4$				*	*	+3	−33	−3	−3	−3		
$a_1^4a_2a_4$				*	*	*	+15	*		+3		
$a_1^6a_4$				*	*	*	*	*	*	−1		
$a_1a_3^3$				−80	+48			+4	−4			
$a_2^2a_3^2$					−32	+2	−12	−3	+3	−1	+1	
$a_1^2a_2a_3^2$						−2	+22	−8	+24	+2	−2	
$a_1a_2^3a_3$								+10	−34	+2	−6	
$a_1^4a_3^2$							−10	*	−16	−1	+1	
$a_1^3a_2^2a_3$								*	+24	−4	+16	
$a_1^5a_2a_3$								*	*	+2	−14	
$a_1^7a_3$								*	*	*	+4	
a_2^5								−3	+12	−1	+4	+1
$a_1^2a_2^4$									−9	+2	−11	−5
$a_1^4a_2^3$										−1	+10	+10
$a_1^6a_2^2$											−3	−10
$a_1^8a_2$												+5
a_1^{10}												−1

TABLE 10B.

a_{10}	+2											
a_1a_9	−2											
a_2a_8	+2	+28										
a_3a_7	−2	−42	+42									
a_4a_6	+2	+48	−48	+32								
a_5^2	−1	−25	+25	−25	+25							
$a_1^2a_8$		−14	*	*	*							
$a_1a_2a_7$		+14	−42	*	*							
$a_1a_3a_6$		−6	+6	−32	*							
$a_2^2a_6$		−4	+32	+16	*	+20						
$a_1a_4a_5$		+2	−2	+18	−50							
$a_2a_3a_5$		+4	−32	−10	+10	−30	+60					
$a_2a_4^2$		−2	+16		+32	+16	−32	+24				
$a_3^2a_4$				+2	+18			+18	+18			
$a_1^3a_7$			+14	*	*	*	*	*	*			
$a_1^2a_2a_6$			−14		*	−20	*	*	*			
$a_1^2a_3a_5$			+14	+12	+20	+15	−30	*	*			
$a_1a_2^2a_5$				−6	−10	+10	−60	*	*			
$a_1^2a_4^2$			−7	−9	+9	−8	+16	−12				
$a_1^4a_6$				*	*	+5	*	*	*			
$a_1^3a_2a_5$				*	*	−5		*	*			
$a_1a_2a_3a_4$				+6	−38	−2	+4	−12	−36			
$a_1^3a_2a_5$				*	*		+50	*	*			
$a_1^5a_5$				*	*	*	−10	*	*			
$a_2^3a_4$					+16	−4	+48	+8	+16	+48		
$a_1^3a_3a_4$				*	*	+1	−2	+12	+12			
$a_1^2a_2^2a_4$				*	*	+2	−44	−6	−6	−72		
$a_1^4a_2a_4$				*	*	*	+10	*		+36		
$a_1^6a_4$				*	*	*	*	*	*	−6		
$a_1a_3^3$				+2	+18			+18	−18			
$a_2^2a_3^2$					−8	+2	−24	−9	+9	−36	+36	
$a_1^2a_2a_3^2$						−1	+22	−12	+36	+36	−36	
$a_1a_2^3a_3$								+10	−34	+24	−72	
$a_1^4a_3^2$							−5	*	−12	−9	+9	
$a_1^3a_2^2a_3$								*	+12	−24	+96	
$a_1^5a_2a_3$								*	*	+6	−42	
$a_1^7a_3$								*	*	*	+6	
a_2^5								−2	+8	−8	+32	+32
$a_1^2a_2^4$									−3	+8	−44	−80
$a_1^4a_2^3$										−2	+20	+80
$a_1^6a_2^2$											−3	−40
$a_1^8a_2$												+10
a_1^{10}												−1

TABLE 11A.

a_{11}	+1											
a_1a_{10}	−11											
a_2a_9	+35	+2										
$a_1^2a_9$	+20	−2										
a_3a_8	−75	−9	+1									
$a_1a_2a_8$	−90	+9	−3									
$a_1^3a_8$	*	*	+2									
a_4a_7	+90	+14	−2	+1								
$a_1a_3a_7$	+240	+16		−4								
$a_2^2a_7$		−30	+10	+3	+16							
$a_1^2a_2a_7$	*	*	−8		−32							
$a_1^4a_7$	*	*	*	*	+16							
a_5a_6	−42	−7	+1	−1	−1	+1						
$a_1a_4a_6$	−420	−63	+9	−2	+5	−5						
$a_2a_3a_6$		+70	−26	−2	−58	+2	+1					
$a_1^2a_3a_6$	*	*	+8	+20	+48	+8	−1					
$a_1a_2^2a_6$	*	*	+8	−15	+62	−6	−3					
$a_1^3a_2a_6$	*	*	*	*	−56		+5					
$a_1a_5^2$	+252	+42	−6	+6	+6	−6						
$a_2a_4a_5$		−21	+7	−6	−5	+45		+1				
$a_1^2a_4a_5$		*	−4	−18	−40			−1				
$a_3^2a_5$		−56	+24	+10	+100	−100	−3	−3	+1			
$a_1a_2a_3a_5$		*	−16	−24	−232	+408	+12	+14	−6			
$a_1^3a_3a_5$		*	*	*	+112	−288	−6	−8	+4			
$a_2^3a_5$		*		+27	+54	−270		−9	+4	+12		
$a_3a_4^2$		+35	−15	−5	−60	+60	+2	+2	−1	−1	+1	
$a_1a_2a_4^2$			+10	+45	+205	−405	−6	−11	+3	+3	−3	
$a_1a_3^2a_4$				−10	−20	+20	−1	−1	*	*	*	
$a_1^5a_6$				*	*	*	−2	*	*	*	*	
$a_1a_3^2a_4$				*	*	*			+3	+8	−8	
$a_2^2a_3a_4$				−45	−90	+450	−3	+14	−6	−36	+6	+1

TABLE 11A (*Continued*).

$a_2a_3^3$	+20	+40	−200	+2	−6	+2	+18		−1	+1	
$a_1^3a_4^2$		−70	+270	+4	+9	−2	−2	+2		*	
$a_1^2a_2^2a_5$			+216	−9	+6	−3	−36	*	*	*	
$a_1^2a_2a_3a_4$			−360	+3	−16	+6	+36	+24	−2	*	
$a_1^2a_3^3$			+160		+8	−8	−34	+16	+1	−1	
$a_1a_2^3a_4$				+9	+3	−2	+48	−18	−3	*	
$a_1a_2^2a_3^2$				−6	−2	+6	−18	−24	+5	−9	
$a_2^4a_3$						−1	−3	+9	−1	+4	+1
$a_1^4a_2a_5$				+6			+36	*	*	*	*
$a_1^4a_3a_4$				−2		−4	−14	−16	+1	*	*
$a_1^3a_2^2a_4$				−6		+3	−72	+12	+8	*	*
$a_1^3a_2a_3^2$				+4		+8	+78	−48	−7	+15	
$a_1^2a_2^3a_3$						−10	−30	+72	−5	+11	−4
$a_1a_2^5$						+3	+9	−27	+3	−12	−3
$a_1^6a_5$							−12	*	*	*	*
$a_1^5a_2a_4$							+30		−7	*	*
$a_1^5a_3^2$							−20	+32	+2	−6	
$a_1^4a_2^2a_3$								−48	+10	−39	+6
$a_1^3a_2^4$								+18	−5	+29	+14
$a_1^7a_4$									+2	*	*
$a_1^6a_2a_3$									−4	+32	−4
$a_1^5a_2^3$									+2	−23	−26
$a_1^8a_3$										−8	+1
$a_1^7a_2^2$										+6	+24
$a_1^9a_2$											−11
a_1^{11}											+2

Table 11B.

a_{11}	+11											
a_1a_{10}	−11											
a_2a_9	+7	+12										
$a_1^2a_9$	+2	−6										
a_3a_8	−5	−18	+42									
$a_1a_2a_8$	−2	+6	−42									
$a_1^3a_8$	*	*	+14									
a_4a_7	+3	+14	−42	+14								
$a_1a_3a_7$	+2	+4		−14								
$a_2^2a_7$		−5	+35	+7	+56							
$a_1^2a_2a_7$	*	*	−14		−56							
$a_1^4a_7$	*	*	*	*	+14							
a_5a_6	−1	−5	+15	−10	−15	+5						
$a_1a_4a_6$	−2	−9	+27	−4	+15	−5						
$a_2a_3a_6$		+5	−39	−2	−87	+1	+30					
$a_1^2a_3a_6$	*	*	+6	+10	+36	+2	−15					
$a_1a_2^2a_6$	*	*	+4	−5	+31	−1	−30					
$a_1^3a_2a_6$	*	*	*	*	−14		+25					
$a_1a_5^2$	+1	+5	−15	+10	+15	−5						
$a_2a_4a_5$		−1	+7	−4	−5	+15		+20				
$a_1^2a_4a_5$		*	−2	−6	−20			−10				
$a_3^2a_5$		−2	+18	+5	+75	−25	−45	−45	+45			
$a_1a_2a_3a_5$		*	−4	−4	−58	+34	+60	+70	−90			
$a_1^3a_3a_5$		*	*	*	+14	−12	−15	−20	+30			
$a_2^3a_5$		*		+3	+9	−15		−30	+40	+80		
$a_3a_4^2$		+1	−9	−2	−36	+12	+24	+24	−36	−24	+12	
$a_1a_2a_4^2$			+2	+6	+41	−27	−24	−44	+36	+24	−12	
$a_1a_3^2a_4$				−1	−3	+1	−3	−3		*	*	
$a_1^5a_6$				*	*	*	−5	*		*	*	
$a_1a_3^2a_4$				*	*	*			+27	+48	−24	
$a_2^2a_3a_4$				−3	−8	+15	−6	+28	−36	−144	+12	+72

Table 11B (*Continued*).

$a_2a_3^3$	+1	+3	−5	+3	−9	+9	+54		−54	+54	
$a_1^3a_4^2$		−7	+9	+8	+18	−12	−8	+4		*	
$a_1^2a_2^2a_5$			+6	−15	+10	−15	−120	*	*	*	
$a_1^2a_2a_3a_4$			−6	+3	−16	+18	+72	+24	−72	*	
$a_1^2a_3^3$			+2		+6	−18	−51	+12	+27	−27	
$a_1a_2^3a_4$				+6	+2	−4	+64	−12	−72		
$a_1a_2^2a_3^2$				−3	−1	+9	−18	−12	+90	−162	
$a_2^4a_3$						−1	−2	+3	−12	+48	+48
$a_1^4a_2a_5$				+5			+60	*	*	*	*
$a_1^4a_3a_4$				−1		−6	−14	−8	+18	*	*
$a_1^3a_2^2a_4$				−2		+3	−48	+4	+96	*	*
$a_1^3a_2a_3^2$				+1		+6	+39	−12	−63	+135	
$a_1^2a_2^3a_3$						−5	−10	+12	−30	+66	−96
$a_1a_2^5$						+1	+2	−3	+12	−48	−48
$a_1^6a_5$							−10	*	*	*	*
$a_1^5a_2a_4$							+10		−42	*	*
$a_1^5a_3^2$							−5	+4	+9	−27	
$a_1^4a_2^2a_3$								−4	+30	−117	+72
$a_1^3a_2^4$								+1	−10	+58	+112
$a_1^7a_4$									+6	*	*
$a_1^6a_2a_3$									−6	+48	−24
$a_1^5a_2^3$									+2	−23	−104
$a_1^8a_3$										−6	+3
$a_1^7a_2^2$										+3	+48
$a_1^9a_2$											−11
a_1^{11}											+1

TABLE 12A.

a_{12}	+1															
a_1a_{11}	−12															
a_2a_{10}	+66	+3														
a_3a_9	−220	−15	+15													
a_4a_8	+495	+40	−40	+1												
a_5a_7	−792	−70	+70	−4	25											
a_6^2	+462	+42	−42	+3	−24	+1										
$a_1^2a_{10}$		−3	*	*	*	*										
$a_1a_2a_9$		+15	−45	*	*	*										
$a_1a_3a_8$		−25	+25	−4	*	*										
$a_2^2a_8$		−15	+150	+3	*	*	+1									
$a_1a_4a_7$		+30	−30	+12	−125	*										
$a_2a_3a_7$		+40	−400	−8	+50	*	−4	+4								
$a_1a_5a_6$		−14	+18	−8	+113	−12										
$a_2a_4a_6$		−70	+700	−22	+680	−70	+8	−8	+1							
$a_3^2a_6$				+24	−570	+80			−1	+5						
$a_2a_5^2$		+42	−420	+24	−675	+100	−5	+5	−1	+2	+1					
$a_3a_4a_5$				−36	+925	−200			+2	−19	−2	+18				
a_4^3				+15	−400	+100			−1	+12	+1	−17	+1			
$a_1^3a_9$			+30		*	*	*	*	*	*	*	*	*			
$a_1^2a_2a_8$			−135		*	*	−2	*	*	*	*	*	*			
$a_1^2a_3a_7$			+360		+200	*	+4	−4	*	*	*	*	*			
$a_1a_2^2a_7$					−150	*	+4	−12	*	*	*	*	*			
$a_1^2a_4a_6$			−630		−525	+100	−8	+8	−1		*	*	*			
$a_1a_2a_3a_6$					+350	−200	−4	+4	+2	−30	*	*	*			
$a_2^3a_6$						+100	−4	+32	−1	+20	*	*	*	+1		
$a_1^2a_5^2$			+378		+336	−64	+5	−5	+1	−2	−1		*			
$a_1a_2a_4a_5$					−105	+20	+2	−2	−2	+37	−4	−54	*			
$a_1a_3^2a_5$					−280	+320			−2	+46	+8	−72	*			
$a_2^2a_3a_5$						−200	+8	−64	+2	−49	−2	+54	*	−3	+3	
$a_1a_3a_4^2$					+175	−200			+2	−41	−2	+114	−12			
$a_1^4a_8$						*	+1	*	*	*	*	*	*	*	*	
$a_1^3a_2a_7$						*	−4	+20	*	*	*	*	*	*	*	
$a_1^3a_3a_6$						*	+4	−4		+20	*	*	*		*	
$a_1^2a_2^2a_6$						*	+4	−60		−15	*	*	*	−3	*	
$a_1^3a_4a_5$						*	−2	+2		−18	+6	+36	*			
$a_1^5a_7$						*	*	−8	*	*	*	*	*	*	*	
$a_1^2a_2a_3a_5$						*	−8	+120	*	*	*	+216	*	*	*	
$a_1^4a_2a_6$						*	*	+28	*	*	*	*	*	*	*	
$a_2^2a_4^2$						+125	−5	+40	+1	−32	+14	+162	−18	+2	−2	+1
$a_1^4a_2a_6$							*	*	*	*	*	*	*	+3	*	*
$a_1^2a_2a_3a_5$							*	*	*	−24	−12	*	*	+6	−6	*

$a_1^6a_6$	*	*	*	*	*	*	*	−1	*	*					
$a_1a_2^3a_5$			*	+27	+6	−162		+3	−9	*					
$a_1^4a_3a_5$	*	−56	*	*	*	−144		−3	+3	*					
$a_1^3a_2^2a_5$	*		*	*		+108		−6	+24	*					
$a_1^5a_2a_5$	*	*	*	*	*	*	*	+3	−21	*					
$a_1^7a_5$	*	*	*	*	*	*	*	*	+6	*					
$a_2a_3^2a_4$			−3	+91	−22	−342	+54			−2	+1				
$a_1^2a_2a_4^2$	+5	−75	*	+45	−15	−360	+54	−4	+4	−2					
$a_1^2a_3^2a_4$			*	−10	+10	+66	−6			+2	−1				
$a_1a_2^2a_3a_4$			*	−45	+30	+486	−180	−1	+1	+4	−6				
$a_2^4a_4$			*		−15	−81	+81	−3	+18	−2	+4		+1		
$a_1^4a_4^2$		+35	*	*	*	+135	−27	+2	−2	+1					
$a_1^3a_2a_3a_4$			*	*	*	−180	+108	+2	−2	−4	+10				
$a_1^2a_2^3a_4$			*	*	*		−54	+6	−51	+2	−7		−4		
$a_1^5a_3a_4$			*	*	*	*	*	−1	+1	*	−4				
$a_1^4a_2^2a_4$			*	*	*	*	*	−3	+48	*	+3		+6		
$a_1^6a_2a_4$			*	*	*	*	*	*	−15	*	*		−4		
$a_1^8a_4$			*	*	*	*	*	*	*	*	*		+1		
a_3^4			+1	−32	+9	+135	−27			+1	−1	+1			
$a_1a_2a_3^3$				+20	−20	−252	+108			−4	+8	−12			
$a_2^3a_3^2$					+10	+54	−54	+2	−12	+2	−5	+8	−1	+1	
$a_1^3a_3^3$						+80	−64				−4	+8			
$a_1^2a_2^2a_3^2$							+36	−4	+34	+4	−9	+30	+3	−3	
$a_1a_2^4a_3$										−4	+14	−48	+2	−6	
$a_1^4a_2a_3^2$								+2	−32	*	+8	−48	−3	+3	
$a_1^3a_2^3a_3$										*	−10	+68	−6	+22	
$a_1^6a_3^2$									+10	*	*	+16	+1	−1	
$a_1^5a_2^2a_3$										*	*	−24	+6	−30	
$a_1^7a_2a_3$										*	*	*	−2	+18	
$a_1^9a_3$										*	*	*	*	−4	
a_2^6										+1	−4	+16	−1	+4	+1
$a_1^2a_2^5$											+3	−24	+3	−15	−6
$a_1^4a_2^4$												+9	−3	+21	+15
$a_1^6a_2^3$													+1	−13	−20
$a_1^8a_2^2$														+3	+15
$a_1^{10}a_2$															−6
a_1^{12}															+1

TABLE 12A (*Continued*).

Table 12B.

a_{12}	+2															
a_1a_{11}	−2															
a_2a_{10}	+2	+18														
a_3a_9	−2	−27	+378													
a_4a_8	+2	+82	−428	+28												
a_5a_7	−2	−35	+490	−70	+525											
a_6^2	+1	+18	−252	+45	−432	+9										
$a_1^2a_{10}$		−9	*	*	*	*										
$a_1a_2a_9$		+9	−378	*	*	*										
$a_1a_3a_8$		−5	+70	−28	*	*										
$a_2^2a_8$		−2	+280	+28	*	*	+56									
$a_1a_4a_7$		+3	−42	+42	−525	*										
$a_2a_3a_7$		+2	−280	−14	+175	*	−21	+84								
$a_1a_5a_6$		−1	+18	−20	+339	−18										
$a_2a_4a_6$		−2	+280	−22	+816	−42	+96	−96	+240							
$a_3^2a_6$				+18	−513	+36			−180	+225						
$a_2a_5^2$		+1	−140	+20	−675	+50	−50	+50	−200	+100	+200					
$a_3a_4a_5$				−18	+555	−60			+240	−570	−240	+720				
a_4^3				+6	−192	+24			−96	+288	+96	−544	+32			
$a_1^3a_9$			+126		*	*	*	*	*	*	*	*	*			
$a_1^2a_2a_8$			−126		*	*	−56	*	*	*	*	*	*			
$a_1^2a_3a_7$			+126		+210	*	+168	−42	*	*	*	*	*			
$a_1a_2^2a_7$					−105	*	+28	−84	*	*	*	*	*			
$a_1^2a_4a_6$			−126		−315	+30	−48	+48	−120		*	*	*			
$a_1a_2a_3a_6$					+105	−30	−12	+12	+120	−450	*	*	*			
$a_2^3a_6$						+10	−8	+64	−40	+200	*	*	*	+40		
$a_1^2a_5^2$			+63		+168	−16	+25	−25	+100	−50	−100		*			
$a_1a_2a_4a_5$					−21	+2	+4	−4	−80	+370	−160	−720	*			
$a_1a_3^2a_5$					−42	+24			−60	+345	+240	−720	*			
$a_2^2a_3a_5$						−10	+8	−64	+40	−245	−40	+360	*	−60	+120	
$a_1a_3a_4^2$					+21	−12			+48	−246	−48	+912	−96			
$a_1^4a_8$						*	+14	*	*	*	*	*	*	*	*	
$a_1^3a_2a_7$						*	−14	+70	*	*	*	*	*	*	*	
$a_1^3a_3a_6$						*	+6	−6		+530	*	*	*		*	
$a_1^2a_2^2a_6$						*	+4	−60		−75	*	*	*	−60	*	
$a_1^3a_4a_5$						*	−2	+2		−90	+120	+240	*			
$a_1^5a_7$						*	*	−14	*	*	*	*	*	*	*	
$a_1^2a_2a_3a_5$						*	−4	+60	*	*	*	+720	*	*	*	
$a_1^4a_2a_6$						*	*	+14	*	*	*	*	*	*	*	
$a_2^2a_4^2$						+5	−4	+32	+16	−128	+224	+864	−96	+32	−64	+144
$a_1^4a_2a_6$							*	*	*	*	*	*	*	+30	*	*
$a_1^2a_2a_3a_5$							*	*	*	−60	−120	*	*	+60	−120	*

$a_1^6a_6$	*	*	*	*	*	*	*	−5	*	*					
$a_1a_2^3a_5$			*	+45	+40	−360		+20	−120	*					
$a_1^4a_3a_5$	*	−14	*	*	*	−240		−15	+30	*					
$a_1^3a_2^2a_5$	*		*	*		+120		−20	+160	*					
$a_1^5a_2a_5$	*	*	*	*	*	*	*	+5	−70	*					
$a_1^7a_5$	*	*	*	*	*	*	*	*	+10	*					
$a_2a_3^2a_4$			−36	+273	−132	−1368	+216			−216	+108				
$a_1^2a_2a_4^2$	+2	−30	*	+90	−120	−960	+144	−32	+64	−144					
$a_1^2a_3^2a_4$			*	−15	+60	+132	−12			+108	−54				
$a_1a_2^2a_3a_4$			*	−45	+120	+648	−240	−4	+8	+144	−216				
$a_2^4a_4$			*		−40	−72	+72	−8	+96	−48	+96		+96		
$a_1^4a_4^2$		+7	*	*	*	+180	−36	+8	−16	+36					
$a_1^3a_2a_3a_4$			*	*	*	−120	+72	+4	−8	−72	+180				
$a_1^2a_2^3a_4$			*	*	*		−24	+8	−136	+24	−84		−192		
$a_1^5a_3a_4$			*	*	*	*	*	−1	+2	*	−36				
$a_1^4a_2^2a_4$			*	*	*	*	*	−2	+64	*	+18		+144		
$a_1^6a_2a_4$			*	*	*	*	*	*	−10	*	*		−48		
$a_1^8a_4$			*	*	*	*	*	*	*	*	*		+6		
a_3^4			+9	−72	+81	+405	−81			+81	−81	+81			
$a_1a_2a_3^3$				+15	−60	−252	+108			−108	+216	−324			
$a_2^3a_3^2$					+20	+36	−36	+4	−48	+36	−90	+144	−72	+72	
$a_1^3a_3^3$						+40	−32				−54	+108			
$a_1^2a_2^2a_3^2$							+12	−4	+68	+36	−81	+270	+108	−108	
$a_1a_2^4a_3$										−24	+84	−288	+48	−144	
$a_1^4a_2a_3^2$								+1	−32	*	+36	−216	−54	+54	
$a_1^3a_2^3a_3$										*	−30	+204	−72	+264	
$a_1^6a_3^2$									+5	*	*	+36	+9	−9	
$a_1^5a_2^2a_3$										*	*	−36	+36	−180	
$a_1^7a_2a_3$										*	*	*	−6	+54	
$a_1^9a_3$										*	*	*	*	−6	
a_2^6										+4	−16	+64	−16	+64	+64
$a_1^2a_2^5$											+6	−48	+24	−120	−192
$a_1^4a_2^4$												+9	−12	+84	+240
$a_1^6a_2^3$													+2	−26	−160
$a_1^8a_2^2$														+3	+60
$a_1^{10}a_2$															−12
a_1^{12}															+1

TABLE 12B (*Continued*).

Compound Determinants.

By C. A. Van Velzer, *University of Wisconsin.*

If a determinant of the order h, say A_h, be bordered by p rows and columns, thus making a determinant of the order $h+p$, then the rows and columns forming the border we will call a *gnomon of the order* p, and the determinant of the order $h+p$, obtained by putting on A_h a gnomon of the order p, we will call a p^{th} *gnomonic of* A_h or a *gnomonic of* A_h of the order $h+p$, and the principal diagonal of the square array at the intersection of the rows and columns forming the gnomon we will call the *principal diagonal of the gnomon.*

If from a determinant A_h of the order h we strike out p rows and columns leaving a minor of the order $h-p$, the rows and columns struck out will be called an *aphaereton* (the part which is taken away) *of the order* p, and the principal diagonal of the square array at the intersection of the rows and columns forming the aphaereton will be called the *principal diagonal of the aphaereton.*

It may happen that the very same thing will be designated by the name gnomon at one time and aphaereton at another time, but the point of view is different in the two cases; thus in Fig. 1, $dcgfbe$ is a gnomon considered in reference to ab as a given determinant, but an aphaereton considered in reference to ac as a given determinant.

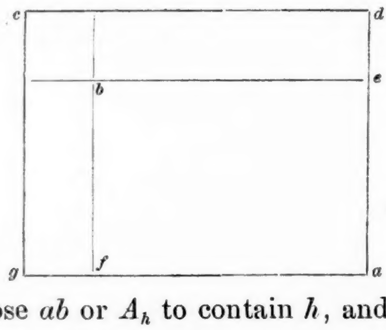


Fig. 1.

Now in Fig. 1, suppose ab or A_h to contain h, and bc to contain s rows and columns, so that ac contains $h+s\ (=n)$ rows and columns. If we form a gnomon of A_h of the order p from p of the rows and columns intersecting in bc, and another gnomon of the order $q(=s-p)$ from the q remaining rows

and columns intersecting in bc, these two gnomons will be called *complementary* gnomons.

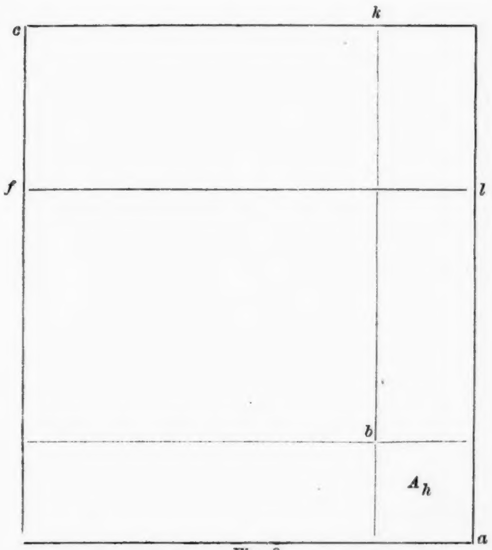


Fig. 2.

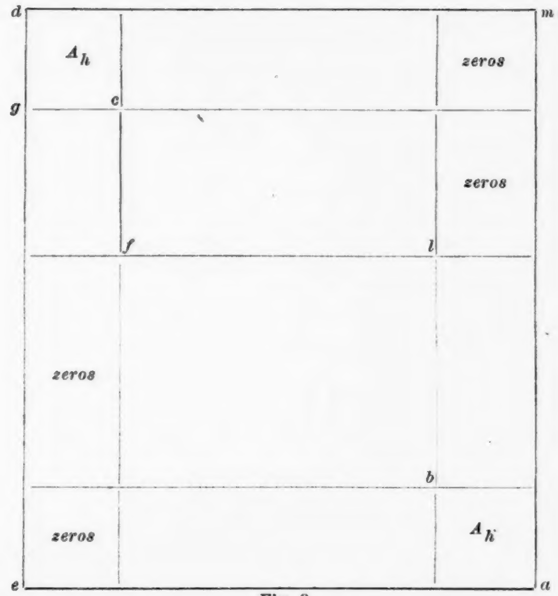


Fig. 3.

In Fig. 2, let ac be a determinant of the order n, say A_n, and select a minor ab or A_h of the order h. From this determinant ac we can form another as follows. Rewrite the last h rows of ac above the first row and the last h columns before the first column, then fill out the vacant upper left-hand corner with the determinant A_h, and finally change into zeros all the elements found at the same time in any of the first $h + p$ rows and the last h columns (*i. e.* in the rectangle lm in Fig. 3), and also all those elements found at the same time in any of the lower $n - p$ rows and the first h columns (*i. e.* in the rectangle ef in Fig. 3).

We have now the determinant ad, Fig. 3, of the order $n + h$, and if we add the first column to the $(n+1)^{\text{th}}$, the second to the $(n+2)^{\text{th}}$, etc., and then subtract the $(n+1)^{\text{th}}$ row from the first row, the $(n+2)^{\text{th}}$ from the second, etc., it is evident that the determinant $ad = A_h . A_n$, for it will then be the determinant A_n bordered by h rows and columns of which the square cd is the determinant, A_h and all the elements in the rectangle cm are zeros.

But the determinant ad, before any combination of rows and columns, is equal to the sum of all minors of the order $h + p$ formed from a given selection of $h + p$ rows each into its complementary minor. As a fixed selection of rows take the first $h + p$ rows, *i. e.* all those above the line fl, Fig. 3. Now any one

of these minors of the order $h+p$ which does not contain all the first h columns will have for its complementary minor a determinant containing a column of zeros, and therefore the product of the two minors must vanish. Also any one of these minors of the order $h+p$ which contains either of the last h columns will vanish from containing a column of zeros. Therefore, in applying Laplace's theorem to this case, the only significant terms are those wherein minors formed from the first $h+p$ rows *necessarily* contain all the first h columns and the complementary minors *necessarily* contain all the last h columns. The sign being properly determined,* all these latter described minors of the order $n-p$ are gnomonics of A_h of the order $n-p$, wherein each gnomon contains all the rows of the rectangle bf whether we consider Fig. 2 or Fig. 3.

Now consider any one of the non-vanishing minors of the order $h+p$ formed above the line fl, Fig. 3, and in that minor pass the first h columns bodily over the remaining p columns and then the first h rows bodily over the remaining p rows, and we have a determinant obtainable from Fig. 2 by putting upon A_h a gnomon of the p^{th} order whose rows and columns intersect in fk.†

From this we draw the important conclusion that if we form all possible gnomonics of A_h of the order $h+p$, wherein the gnomons used are formed from the rows and columns intersecting in fk, and each of these gnomonics be multiplied by the gnomonic of the order $n-p$ formed by putting upon A_h the complementary gnomon, the sum of all these products $= A_h . A_n$.

If one or more of the rows below the line fl should be identical with one or more of the rows above the line fl, then the sum of the products of determinants formed as above will evidently vanish; but we can arrive at this same set of determinants without supposing any two of the rows of A_n identical as follows. Border A_h in every possible way with gnomons of the order p from a fixed selection of p rows, and also in every possible way from a fixed selection of $(n-h-p)$ rows wherein the second fixed selection contains one or more rows of the first selection.

Therefore if we form all possible gnomonics of A_h of the order $h+p$ and also all possible gnomonics of A_h of the order $n-p$ wherein the gnomons used

*Consider p^{th} and $(n-h-p)^{\text{th}}$ gnomonics of A_h formed by putting upon A_h complementary gnomons, then to all p^{th} gnomonics or gnomonics of the order $h+p$ give the sign $+$ and to all $(n-h-p)^{\text{th}}$ gnomonics give the sign $+$ or $-$ according as the principal diagonal of the gnomon used in forming the gnomonic into the principal diagonal of its complementary gnomon gives a positive or negative term of bc.

† This would not be the case if it were not for the fact that the elements in fg, Fig. 3, are the same as those in lk, in Fig. 2.

in forming the first set of gnomonics are taken from a fixed selection of p rows and those used in forming the second set of gnomonics are taken from a fixed selection of $n-h-p$ rows, which selection contains one or more rows of the first selection, and if each gnomonic of the first set is multiplied by that one of the second set in which the *gnomons* used in forming the two gnomonics have no two *columns* the same; then the sum of all these products must be zero.

Let us now form a compound determinant whose elements are p^{th} gnomonics of A_h, the various gnomons being taken from the rows and columns intersecting in bc. Suppose the elements of this determinant are denoted by $p_{r,s}$ and the whole determinant by P.

Also form another compound determinant whose elements are $(n-h-p)^{\text{th}}$ gnomonics of A_h, the gnomons being the complements of those used in forming the *corresponding* elements of P. Denote the elements of this determinant by $q_{r,s}$ and the whole determinant by Q.

From what has been shown above, it follows that P and Q are so related that when the elements of a row of P are multiplied by the corresponding elements of the *same* row of Q, the sum of the products $= A_h . A_n$ or $\sum^s p_{r,s} . q_{r,s} = A_h . A_n$, and if the elements of any row of P be multiplied by the corresponding elements of a *different* row of Q, the sum of the products $= 0$ or $\sum^s p_{r,s} . q_{r',s} = 0$.

Determinants possessing these two properties I shall call reciprocal. Evidently this definition includes the ordinary reciprocal determinants formed from complementary minors of a given determinant.

P and Q are therefore reciprocal determinants, and when multiplied together by the ordinary rule it follows at once that

$$P . Q = A_h^{(n-h)_p} . A_n^{(n-h)_p},$$

where $(n-h)_p$ stands for $\dfrac{(n-h)!}{p!\,(n-h-p)!}$ or the number of combinations of $n-h$ things taken p at a time.

From this value for the product $P . Q$ it follows that

$$P = x . A_h^s . A_n^{s'} \quad \text{and}$$

$$Q = \frac{1}{x} . A_h^{(n-h)_p - s} . A_n^{(n-h)_p - s'}.$$

Now when p changes into $n-h-p$, P becomes Q.* Therefore, whatever x is

* Q has the same value as if all its signs were $+$, for in every row the signs are the same as in the first row or every sign is reversed, and if we change the signs of the elements in every row which begins with a $-$ sign, then the signs in every row are the same as in the first row, and then changing the signs of all the columns which have $-$ signs, we make the same number of changes of sign of columns as we before made of rows, since originally the number of $-$ signs in the first row were the same as in the first column. The determinant has thus been multiplied by -1 an even number of times and the final determinant has all its signs $+$.

it must change into its reciprocal when $n-h-p$ is written for p. Also s and s' change into $(n-h)_p - s$ and $(n-h)_p - s'$ respectively. Moreover, when $h=0$, P becomes the determinant of minors of the p^{th} order formed from A_n and we know that then $P = A_n^{(n-1)_{p-1}}$, and when $p=1$ it can easily be proved* that then $P = A_h^{n-h-1}.A_n$, and when $h=n-1$, $P=A_n$.

From all these results we are led to write $x=1$, $s=(n-h-1)_p$, and $s'=(n-h-1)_{p-1}$.†

Hence $$P = A_h^{(n-h-1)_p}.A_n^{(n-h-1)_{p-1}}$$
and $$Q = A_h^{(n-h-1)_{p-1}}.A_n^{(n-h-1)_p}$$
since $$(n-h)_p - (n-h-1)_p = (n-h-1)_{p-1}.$$

In exactly the same manner as in ordinary reciprocal determinants it may be shown that if a minor of P of the order m be represented by P_m, we have

$$P_m = A_h^{m-(n-h-1)_{p-1}}.A_n^{m-(n-h-1)_p}.Q_{m'}$$

where $Q_{m'}$ is the complementary minor of Q; and if a minor of Q of the order m be represented by Q_m, we have

$$Q_m = A_h^{m-(n-h-1)_p}.A_n^{m-(n-h-1)_{p-1}}.P_{m'}$$

where $P_{m'}$ is the complementary minor of P. Gnomonics of A_h are minors of A_n, and the determinant above represented by P may be considered as the determinant of minors of A_n of the order $h+p$, which result from striking out, in every possible way, $n-h-p$ of the rows and columns intersecting in bc.

The determinant Q may also be considered as the determinant of minors of A_n, formed by striking out in every possible way p of the rows and columns intersecting in bc.

Now, since P and Q are reciprocal determinants, we have here an extension of the notion of reciprocal determinants, formed from *minors* of a given determinant, and these reduce to the ordinary reciprocal determinants when the striking out of a certain number of rows and columns is done in every possible way *throughout the whole determinant* A_n.

By making $h=0$ in the above expressions for P, Q, P_m, Q_m, and remembering that a determinant of the zero order $=1$, and also that $0!=1$, and

* See page 169.

† This inferring the form of a function from several particular values may be open to some objection as a proof, but it seems to me to be as free from objection as Picquet's method as given in Scott's Determinants, page 65, wherein an equation is assumed true which we know to be true for a *single* value of one of the letters entering the equation. The theorem was discovered by Professor Sylvester, but I have never seen his proof.

therefore $r_0 = 1$, we have the well-known values of these expressions given in most books on the subject.

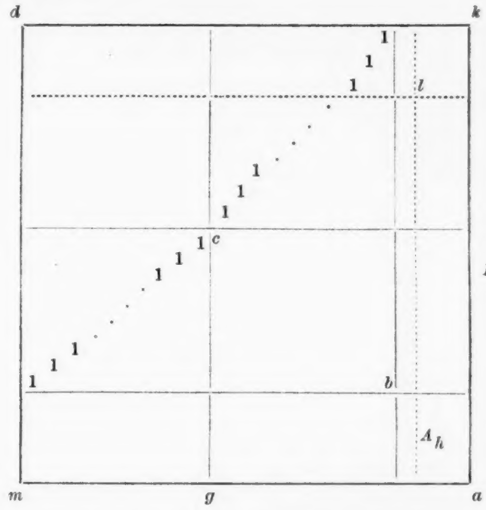


Fig. 4.

From the determinant ac, Fig. 4, of the order n, say A_n, let us form the determinant ad of the order $2n-h$ by bordering ac with $n-h$ rows and columns, as in Fig. 4, wherein all the elements of the gnomon are zeros except those marked 1 in the figure. It is evident that this determinant $ad = \pm$ the determinant ab which is of the order h and will be designated by A_h; the upper or lower sign being used according as $n-h$ is even or odd.

Let us now form the determinant of first minors of ad, and of the resulting determinant take that minor of the order $n-h$ which contains all the elements obtained by striking out in every possible way one of the rows and columns intersecting in cd.

If we call this minor D_{n-h} we have by a well-known theorem (Salmon's Higher Algebra, page 29, Art. 33)

$$D_{n-h} = A_h^{n-h-1} . A_n .$$

But the elements of D_{n-h} are all possible first gnomonics of A_h, where the gnomons used are formed from the rows and columns intersecting in bc, and is the very determinant that we have before called P_m, when $p=1$; and here we have the easy proof spoken of on page 168.

Now we know that if $ad = 0$ the first minors of any row are proportional to those of any other row (Salmon's Higher Algebra, page 29, ex. 1). Therefore if $A_h = 0$ the first gnomonics in any row are proportional to those in any

other row of the determinant of all first gnomonics, or the determinant of all first gnomonics, and also any minor above the first order vanishes.

Instead of first we may form p^{th} minors of ad by striking out p rows and columns, and it is evident that every p^{th} minor formed by removing an aphaereton of the p^{th} order whose rows and columns intersect in cd reduces to a p^{th} gnomonic of A_h, formed by bordering A_h with a gnomon of the p^{th} order whose rows and columns intersect in bc.

Thus we have a means of passing from theorems concerning minors to those concerning gnomonics, but to do so we should always look upon a minor as formed by *taking away an aphaereton* from a given determinant, and not by *selecting a square array* from a given determinant.

It has before been shown that we may pass from theorems concerning gnomonics to those concerning minors, and we have therefore a principle of duality in determinants which may be illustrated by the following dualistic theorems placed side by side.

A determinant P whose elements are p^{th} gnomonics of a determinant A_h of the order h wherein the various gnomons are formed from the rows and columns intersecting in a given determinant A_s of the order s is equal to the original determinant A_h to the $\{(s-1)_p\}^{\text{th}}$ power into the $\{(s-1)_{p-1}\}^{\text{th}}$ power of the determinant obtained by bordering A_h with all the rows and columns intersecting in A_s.	A determinant P' whose elements are p^{th} minors of a determinant A_h of the order h wherein the various aphaeretons are formed from the rows and columns intersecting in a given determinant A_s' of the order s is equal to the original determinant A_h to the $\{(s-1)_p\}^{\text{th}}$ power into the $\{(s-1)_{p-1}\}^{\text{th}}$ power of the determinant obtained by striking out from A_h all the rows and columns intersecting in A_s'.
A minor of P of the order m is equal to the original determinant A_h to the power $m-(s-1)_{p-1}$ into the $\{m-(s-1)_p\}^{\text{th}}$ power of the determinant obtained by bordering A_h with all the rows and columns intersecting in A_s into the complementary minor of the reciprocal determinant.	A minor of P' of the order m is equal to the original determinant A_h to the power $m-(s-1)_{p-1}$ into the $\{m-(s-1)_p\}^{\text{th}}$ power of the determinant obtained by striking out from A_h all the rows and columns intersecting in A_s' into the complementary minor of the reciprocal determinant.
If a determinant vanish the elements in any row of the determinant of first gnomonics are proportional to the elements in any other row.	If a determinant vanish the elements in any row of the determinant of first minors are proportional to the elements in any other row.

Now consider the determinant ad, Fig. 4, divided by the dotted lines in the figure, so that kl is a square array say of the order $k(<h)$. Form two compound determinants whose elements are minors formed by removing aphaeretons of the orders p and $k-p$ respectively, whose rows and columns intersect in kl.

These two determinants are reciprocal, and we may apply to either one the first two theorems given on the right-hand side of the line dividing dualistic theorems. Denote the elements of the first by $p'_{r,s}$, of the second by $q'_{r,s}$, the first compound determinant by P', and the second by Q'. It is evident that the elements $p'_{r,s}$ reduce to those determinants formed by striking out p of the k right-hand columns of ab and substituting in their stead p of the k right-hand columns of bg, while the elements $q'_{r,s}$ reduce to those determinants formed by striking out $k-p$ of the right-hand columns of ab and substituting in their stead $k-p$ of the k right-hand columns of bg, where in forming any element of P' neither striking out nor substituting affects the same columns as in forming the corresponding element of Q'. By applying the first theorem on the right of the line dividing dualistic theorems it follows that

$$P' = (ad)^{(k-1)_p}.(lm)^{(k-1)_{p-1}}.$$

But $ad = A_h$ and lm reduces to the determinant formed by striking out the k right-hand columns of A_h and substituting the k right-hand columns of bg in their place.

Hence the determinant P' equals the original determinant A_h to the power $(k-1)_p$ into the $\{(k-1)_{p-1}\}^{\text{th}}$ power of the determinant produced by striking out and substituting all at once the rows and columns that were struck out and substituted piecemeal in forming the elements of P'.

Now remembering what the elements of P' are we may produce this determinant in another way, as follows: Take two determinants A_h and B_h, placed side by side, and form a determinant whose elements are those determinants obtained by replacing in every possible way p columns from a fixed selection of k columns of A_h, by p columns taken in every possible way from a fixed selection of k columns of B_h. This determinant, P', then equals the original determinant A_h to the power $(k-1)_p$ into the $\{(k-1)_{p-1}\}^{\text{th}}$ power of the determinant produced by replacing the fixed selection of k columns of A_h by the fixed selection of k columns of B_h.

Similarly the determinant Q' equals the determinant B_h to the power $(k-1)_p$ into the $\{(k-1)_{p-1}\}^{\text{th}}$ power of the determinant obtained by replacing the fixed selection of k columns of B_h by the fixed selection of k columns of A_h.

If $k = h$, *i. e.* if the fixed selection of k columns of A_h and B_h is in each case the *whole determinant*, then the determinants P' and Q' are the determinants described in Scott's Determinants, page 56, Art. 4. Moreover, when $k < h$, these determinants P' and Q' are minors of the determinants given in Scott's treatise, but they are here expressed entirely independent of a complementary minor of another determinant.

Applying the second of the theorems on the right of the line dividing dualistic theorems, it readily follows that a minor of P' of the order m is equal to the determinant A_h to the power $m-(k-1)_{p-1}$ into the $\{m-(k-1)_p\}^{\text{th}}$ power of the determinant obtained by replacing the fixed selection of k columns of A_h by the fixed selection of k columns of B_h, into the reciprocal determinant Q'.

P. S.—Since writing the above I have received and read an article on Compound Determinants by Mr. R. F. Scott, published in the Proceedings of the London Mathematical Society, Vol. XIV, page 91.

I was never satisfied with Picquet's proof of the theorem discovered by Professor Sylvester and tried to find some simple and rigorous proof. Failing in this, I wrote out the one given above as one to my own mind preferable to Picquet's, but Mr. Scott's proof, in section 2 of his article, seems to leave nothing to be desired either in simplicity or rigor.

In the determinant which I have called A_n, if $n = 2h$, and if A_n is a determinant formed by putting on A_h a gnomon of the order h, wherein the h^2 elements at the intersection of the rows and columns forming the gnomon are all zeros, then the theorems given by Mr. Scott, in his article from section 5 on, are easily seen to be special cases of some of the theorems given in this paper.

Extract from a Letter to Mr. Sylvester from M. Hermite.

... puisque vous êtes assez bon pour publier mon énoncé sur la fonction qui représente le nombre des décompositions d'un entier en deux carrés j'y ajouterai quelques observations sur une question voisine en considérant la fonction $\phi(n)$ égale à la somme des diviseurs de n. J'ai été amené à joindre à la fonction $E(x)$ qui désigne l'entier contenu dans x les deux suivantes

$$E_1(x) = E(x + \tfrac{1}{2}) - E(x)$$

et

$$E_2(x) = \tfrac{1}{2}[E^2(x) + E(x)]$$

La première qu'on peut aussi exprimer par

$$E_1(x) = E(2x) - 2E(x)$$

et dont Gauss a fait usage, a pour principale propriété que :

$$E_1(x+1) = E_1(x)$$

$$E_1(x + \tfrac{1}{2}) = 1 - E_1(x)$$

et on voit qu'elle est toujours égale à zéro ou à l'unité. Elle a pour valeur l'unité lorsque cette différence égale ou surpasse $\frac{1}{2}$. Cela étant voici une circonstance dans laquelle elle se présente.

Soit $F(n)$ le nombre des représentations de n par la forme $x^2 + y^2$, on aura:

$$F(2) + F(6) + \ldots + F(4n+2) =$$

$$4\left[E_1\left(\frac{2n+1}{2}\right) + E_1\left(\frac{2n+2}{6}\right) + E_1\left(\frac{2n+3}{10}\right) + \ldots E_1\left(\frac{4n+1}{4n+2}\right)\right].$$

Mais c'est surtout de la seconde fonction $E_2(x)$ que je vais m'occuper.

Soit n un entier impair et $\phi(n)$ la somme de ses diviseurs, on aura

$$\phi(1) + \phi(3) + \ldots + \phi(n) =$$

$$E_2\left(\frac{n+1}{2}\right) + E_2\left(\frac{n+3}{6}\right) + E_2\left(\frac{n+5}{10}\right) + \ldots$$

$$+ E_2\left(\frac{n-1}{2}\right) + E_2\left(\frac{n-3}{6}\right) + E_2\left(\frac{n-5}{10}\right) + \ldots$$

les deux sommes étant continuées jusqu'à ce que les quantités sous le signe E_2 deviennent moindres que l'unité.

De cette expression de $\Sigma\phi(n)$ j'ai tiré les conclusions suivantes.

Soit $\lambda = E(\sqrt{n})$; la somme relative aux entieres impaires qui sont $\equiv 3$ mod 4, à savoir $\phi(3)+\phi(7)+\ldots+\phi(4n-1)$ a pour valeur

$$2(n^2+n)+4\sum_{1}^{\lambda}{}_c\left[cE\left(\frac{n-c^2}{2c+1}\right)+E_2\left(\frac{n-c^2}{2c+1}\right)\right].$$

La même somme, en considérant les nombres $\equiv 1$ mod 4, c'est à dire

$$\phi(1)+\phi(5)+\ldots+\phi(4n+1)$$

conduit à considérer deux cas. Je suppose d'abord que $4n+5$ ne soit pas un carré; en faisant

$$\lambda = E\left(\frac{\sqrt{4n+1}+1}{2}\right)$$

j'obtiens

$$2\sum_{1}^{\lambda-1}{}_c E^2\left(\frac{n+c^2}{2c-1}\right)-\frac{2\lambda^3+6\lambda^2+\lambda}{3}.$$

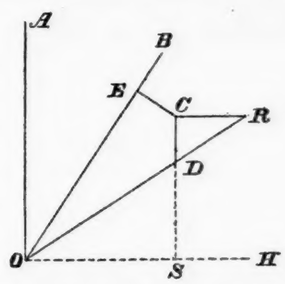
Mais s'il arrive que $4n+5$ soit un carré le terme algébrique se modifie; la valeur de la somme étant dans ce cas

$$2\sum_{1}^{\lambda-1}{}_c E^2\left(\frac{n-c^2}{2c-1}\right)-\frac{2\lambda^3+\lambda}{3}.$$

A Graphic Method of Solving Spherical Triangles.

BY CHARLES H. SMITH, *Professor of Mathematics in Bowdoin College.*

AOB and AOR are respectively equal to the sides c and b of a spherical triangle, OD is the natural cosine of the included angle A, OE is the natural cosine of the third side a, and OR is the radius of the natural tables; DC is



drawn parallel to OA, RC perpendicular to OA, and EC perpendicular to OB. It may readily be shown that these three lines meet in C, which is the orthographic projection of the triangular vertex C on the plane of the side c.

With AOB and AOR given, it is evident that if either OD or OE is known, the other may be found, and if both are known together with AOR, the two values of AOB then possible may be found by drawing from C tangents to an arc which has O as center and the given length of OE as radius.

(The constructions on OB and OR may be exchanged.)

Thus a spherical triangle may be solved when there are given three sides, two sides and included angle, or two sides and angle opposite one of them; and the three remaining cases of the spherical triangle may be solved through the three just named which are polar to them.

When great accuracy is not required, it is believed that the method here sketched will be found useful, on account of its simplicity and the ease with which it may be applied.

Thus, to find the distance between two places whose latitudes and longitudes are known, take HOR and HOB the latitudes and OD the cosine of the difference of longitude, then through D draw SC the sine of HOR and CE perpendicular to OB; OE is the cosine of the great circle arc between the places.

To reduce a sextant-angle to the horizon, take HOR and HOB the angular altitudes of the two points in space, OE the cosine of the measured angle between them, and OR the radius of the natural tables, then draw EC perpendicular to OB, and RC parallel to OH, and CD perpendicular to OH; OD is the cosine of the horizontal angle required.

The construction last mentioned gives also the cosine of the angle between an hour circle through a heavenly body whose declination and altitude are known, and the meridian of a place whose latitude is known. In this case OE is the sine of the altitude, and of course is zero for an observation at the horizon.

Other applications will readily occur.

Note on Weierstrass' Methods in the Theory of Elliptic Functions.

By A. L. Daniels, *Johns Hopkins University.*

The work of Professor Weierstrass in the modern function-theory is of such commanding importance that it may not be out of place to give a clear and elementary account of his somewhat peculiar nomenclature and methods for the benefit of those English readers who have not had the opportunity of listening to his lectures. This is especially desirable in the theory of doubly-periodic functions, where his symbols and methods differ not a little from those of Jacobi and his predecessors. The only connected and systematic statement of Weierstrass' methods in this field is contained in the "*Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen. Nach Vorlesungen und Aufzeichnungen des Herrn K. Weierstrass bearbeitet und herausgegeben von H. A. Schwarz. Göttingen,* 1882." Professor Schwarz has prepared this little work with infinite pains for the use of his own and Weierstrass' students exclusively. I rely in the following chiefly on this work, on a large number of lithographed formulæ prepared by Prof. Schwarz, and on notes of lectures on this topic by Profs. Schwarz and Weierstrass.

We commence with the sigma-function, to designate which Weierstrass employs a slightly altered form of the Greek sigma. The simplest possible analytic function, which for all finite values of the argument retains the character of an integral function, and which becomes an infinitesimal of the first order for $u=0$ and $u=w$, where $w=2\mu\omega+2\mu'\omega'$, is represented by the formula

$$\sigma u = u\Pi'_w\left(1-\frac{u}{w}\right)e^{\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}},$$

where 2ω, $2\omega'$ are the so-called "*periods,*" and μ, μ' assume all real integral values, positive and negative, excepting only the combination $\mu=0$, $\mu'=0$; which exception is indicated by the dash at the right of Π. The exponential factor is to be repeated with each of the factors $1-\dfrac{u}{w}$, and is necessary to the

convergence of the infinite product. For

$$\log \sigma u = \log u + \Sigma' \left\{ \log\left(1 - \frac{u}{w}\right) + \frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2} \right\}$$
$$= \log u - \Sigma' \left\{ \frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2} + \frac{1}{3}\frac{u^3}{w^3} + \ldots - \frac{u}{w} - \frac{1}{2}\frac{u^2}{w^2} \right\};$$

the exponential factor furnishing the last two terms, which are seen at once to be necessary, because $\Sigma \frac{1}{w}$ and $\Sigma \frac{1}{w^2}$ do not converge.

The remaining terms all converge, and the formula becomes

$$\log \sigma u = \log u - \frac{u^3}{3}\Sigma\frac{1}{w^3} - \frac{u^4}{4}\Sigma\frac{1}{w^4} - \ldots$$

The function σu may however be presented in a slightly different form by combining with every pair of values of μ, μ' the corresponding pair with opposite signs, and we have instead of $\Pi\left(1 - \frac{u}{w}\right)$ from $-\infty$ to $+\infty$, $\Pi\left(1 - \frac{u^2}{w^2}\right)$ from 1 to $+\infty$, and the formula becomes

$$\sigma u = u \prod_{1}^{w=+\infty}{}' \left(1 - \frac{u^2}{w^2}\right) e^{\frac{u^2}{w^2}}. \qquad \begin{array}{c} w = 2\mu\omega + 2\mu'\omega', \\ \mu, \mu' = 1, 2 \ldots + \infty. \end{array}$$

Developed in an infinite series

$$\sigma u = u - \frac{u^5}{4}\Sigma\frac{1}{w^4} - \frac{u^7}{6}\Sigma\frac{1}{w^6} - \ldots$$

The coefficients in this series are integral functions of two constants,

$$g_2 = 2^2.3.5\Sigma'_w \frac{1}{w^4}; \quad g_3 = 2^2.5.7.\Sigma'_w \frac{1}{w^6},$$

called the invariants of the corresponding sigma-function, and which are functions of course of the half periods ω, ω'.

The series for σu then takes the form

$$\sigma u = u + * - \frac{g_2 u^5}{2^4.3.5} - \frac{g_3 u^7}{2^3.3.5.7} - \frac{g_2^2 u^9}{2^9.3^2.5.7} - \frac{g_2 g_3 u^{11}}{2^7.3^2.5^2.7.11} - \ldots$$

The sigma function is not an elliptic function, and does not possess an addition-theorem in the usual sense, neither is it periodic; but on increasing the argument by one period, 2ω, we have

$$\sigma(u + 2\omega) = -e^{2\frac{\sigma'\omega}{\sigma\omega}(u+\omega)}.\sigma u$$

and in like manner

$$\sigma(u + 2\omega') = -e^{2\frac{\sigma'\omega'}{\sigma\omega'}(u+\omega')}.\sigma u$$

Weierstrass writes $p\omega + q\omega' = \tilde{\omega}$, $p\eta + q\eta' = \tilde{\eta}$

$$\frac{\sigma'\omega}{\sigma\omega} = \eta, \quad \frac{\sigma'\omega'}{\sigma\omega'} = \eta';$$

so that in general $\sigma(u+2\bar{\omega}) = \mp e^{2\bar{\eta}(u+\bar{\omega})}\sigma u.$

Although σu does not possess an addition theory, the contrary is the case with the second logarithmic derivative $\frac{\sigma'' u}{\sigma u}$, for which Weierstrass has employed again a modified letter

$$\begin{aligned} \wp u &= -\frac{d^2}{du^2}\log \sigma u \\ &= \frac{1}{u^2} + \Sigma'_w\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right) \\ &= \frac{1}{u^2} + * + \frac{g_2}{2^2.5}u^2 + \frac{g_3}{2^2.7}u^4 + \dots \end{aligned}$$

where g_2, g_3 are the invariants of σu before mentioned. It is especially to be noticed that u occurs in only one term with a negative exponent, and that the constant term of the series is null. The function $\wp u$ is an elliptic function of the second degree, and is moreover the simplest possible doubly-periodic function. Among the many interesting relations, we ought to notice that, the half-periods of $\wp u$ being ω and ω', if we write $\omega + \omega' = \omega''$

$$\wp\omega = e_1, \quad \wp\omega'' = e_2, \quad \wp\omega' = e_3,$$

and further

$$(\wp' u)^2 = 4(\wp u - \wp\omega)(\wp u - \wp\omega'')(\wp u - \wp\omega') = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3)$$

$= 4\wp^3 u - g_2\wp u - g_3$, whence $e_1 \times e_2 \times e_3 = 0$, $\Sigma e_\lambda e_\mu = -\frac{1}{4}g_2$, $e_1e_2e_3 = \frac{1}{4}g_3$, or, if we write $\wp u = s$, $\wp u$ appears as the elliptic function corresponding to the integral

$$u = \int \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}},$$

For the sum of two arguments, the function $\wp(u+v)$ is expressible as a rational function of $\wp u$, $\wp v$, $\wp' u$, $\wp' v$, for example,

$$\wp(u \pm v) = \frac{1}{4}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]^2 - \wp u - \wp v.$$

As to the derivatives of $\wp u$; the remarkable fact deserves notice that all the derivatives of an even order are entire functions of $\wp u$ itself. For instance

$$\wp'' u = 6\wp^2 u - \tfrac{1}{2}g_2;\ \wp^{\mathrm{iv}} u = 120\wp^3 u - (12g_2 - 18g_2)\wp u - 12g_3.$$

Another remarkable relation between the functions $\wp$ and σ is easily deduced, namely

$$\wp u - \wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v},$$

which Professor Schwarz is in the habit of calling the "pocket edition" of the elliptic functions. The following proposition is one of great importance in the theory of functions.

If ϕu denotes any elliptic function of the r^{th} degree with the periods 2ω, $2\omega'$, and if σu has the same pair of periods, then we can always determine the $2r+1$ quantities $u_1, u_2, \ldots u_r;\ v_1, v_2, \ldots v_r,\ C$ so that

$$\phi(u) = C.\frac{\sigma(u-u_1)\,\sigma(u-u_2)\ldots\sigma(u-u_r)}{\sigma(u-v_1)\,\sigma(u-v_2)\ldots\sigma(u-v_r)};$$

which proposition is capable of inversion. An analogous theorem in regard to $\wp u$ is, if $u_0, u_1, u_2, \ldots u_n$ denote $n+1$ independent variables, then the function

$$\phi(u_0, u_1, u_2, \ldots u_n) = \begin{vmatrix} 1 & \wp u_0 & \wp' u_0 \ldots \wp^{(n-1)} u_0 \\ 1 & \wp u_1 & \wp' u_1 \ldots \wp^{(n-1)} u_1 \\ 1 & \wp u_2 & \wp' u_2 \ldots \wp^{(n-1)} u_2 \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ 1 & \wp u_n & \wp' u_n \ldots \wp^{(n-1)} u_n \end{vmatrix}$$

is an elliptic function of the degree $n+1$ of any one of the arguments $u_0, u_1 \ldots u_n$. In general "every unique elliptic function $\phi(u)$ is expressible as a rational function of $\wp u$ and the first derivative $\wp' u$ with the same pair of periods 2ω, $2\omega'$ as $\phi(u)$; and in like manner $\wp u$ and $\wp' u$ are expressible as rational functions of ϕu and $\phi' u$".

With the function σu are closely connected the following

$$\sigma_1 u = \frac{e^{-\eta u}\,\sigma(\omega+u)}{\sigma\omega} = \frac{e^{\eta u}\,\sigma(\omega-u)}{\sigma\omega}$$

$$\sigma_2 u = \frac{e^{-\eta'' u}\,\sigma(\omega''+u)}{\sigma\omega''} = \frac{e^{\eta'' u}\,\sigma(\omega''-u)}{\sigma\omega''}$$

$$\sigma_3 u = \frac{e^{-\eta' u}\,\sigma(\omega'+u)}{\sigma\omega'} = \frac{e^{\eta' u}\,\sigma(\omega'-u)}{\sigma\omega'}$$

where ω, ω' are the half periods, and $\omega+\omega'=\omega''$, $\dfrac{\sigma'\omega}{\sigma\omega}=\eta$, $\dfrac{\sigma'\omega'}{\sigma\omega'}=\eta'$, $\eta+\eta'=\eta''$. By inserting in the "pocket edition" for v the values respectively ω, ω'', ω', we have

$$\wp u - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2, \quad \wp u - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2, \quad \wp u - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2$$

whereby the following relations are established for the differences of the roots. Remembering that $\wp\omega = e_1$, $\wp\omega''=e_2$, $\wp\omega'=e_3$.

$$\sqrt{e_1-e_2} = \frac{\sigma_2\omega}{\sigma\omega}, \quad \sqrt{e_2-e_3} = \frac{\sigma_3\omega''}{\sigma\omega''}, \quad \sqrt{e_1-e_3} = \frac{\sigma_3\omega}{\sigma\omega}$$

$$\sqrt{e_2-e_1} = \frac{\sigma_1\omega''}{\sigma\omega''}, \quad \sqrt{e_3-e_2} = \frac{\sigma_2\omega'}{\sigma\omega'}, \quad \sqrt{e_3-e_1} = \frac{\sigma_1\omega'}{\sigma\omega'}$$

where we assume $e_1 > e_2 > e_3$. If now we assume $R\left(\dfrac{\omega'}{\omega i}\right) > 0$, that is, the real component of the complex $\dfrac{\omega'}{\omega\sqrt{-1}} > 0$, so that in the geometrical representation

the point ω' lies "above" the right line joining $u=0$ and $u=\omega$, then

$$\sqrt{e_3-e_2}=-i\sqrt{e_2-e_3};\quad \sqrt{e_3-e_1}=-i\sqrt{e_1-e_3};\quad \sqrt{e_2-e_1}=-i\sqrt{e_1-e_2}.$$

If now we denote for convenience by λ, μ, ν the indices 1, 2, 3, and write

$$\frac{\sigma u}{\sigma_\lambda u}=\xi_{o\lambda},\quad \frac{\sigma_\mu u}{\sigma_\nu u}=\xi_{\mu\nu},\quad \frac{\sigma_\lambda u}{\sigma u}=\xi_{\lambda o}, \text{ etc.}$$

remembering that
$$\wp' u=-2\frac{\sigma_\lambda u.\sigma_\mu u.\sigma_\nu u}{\sigma u.\sigma u.\sigma u},$$

we easily obtain

$$\frac{d\xi_{o\lambda}}{du}=\xi_{\mu\lambda}\xi_{\nu\lambda},\quad \frac{d\xi_{\mu\nu}}{du}=-(e_\mu-e_\nu)\xi_{\lambda o}\xi_{o\nu},\quad \frac{d\xi_{\lambda o}}{du}=-\xi_{\mu o}\xi_{\nu o},$$

whence
$$\left(\frac{d\xi_{o\lambda}}{du}\right)^2=(1-(e_\mu-e_\lambda)\xi_{o\lambda}^2)(1-(e_\nu-e_\lambda)\xi_{o\lambda}^2),$$

$$\left(\frac{d\xi_{\mu\nu}}{du}\right)^2=(1-\xi_{\mu\nu}^2)(e_\mu-e_\lambda+(e_\lambda-e_\nu)\xi_{\mu\nu}^2),$$

$$\left(\frac{d\xi_{\lambda o}}{du}\right)^2=(\xi_{\lambda o}^2+e_\lambda-e_\mu)(\xi_{\lambda o}^2+e_\lambda-e_\nu),$$

and the four functions

$$\frac{\sigma u}{\sigma_\lambda u},\quad \frac{1}{\sqrt{e_\mu-e_\lambda}}\frac{\sigma_\mu u}{\sigma_\nu u},\quad \frac{1}{\sqrt{e_\nu-e_\lambda}}\frac{\sigma_\nu u}{\sigma_\mu u},\quad \frac{1}{\sqrt{e_\mu-e_\lambda}\sqrt{e_\nu-e_\lambda}}\frac{\sigma_\lambda u}{\sigma u}$$

satisfy the same differential equation

$$\left(\frac{d\xi}{du}\right)^2=(1-(e_\mu-e_\lambda)\xi_2)(1-(e_\nu-e_\lambda)\xi^2).$$

But the English reader will desire to know in what connection the system of Weierstrass stands to the more widely known systems of Jacobi and Legendre. If we define the k of Jacobi by the equation

$$k^2=\frac{e_2-e_3}{e_1-e_3},$$

then the following relations are established between the sigma-quotients and Jacobi's functions. We give only three as specimens, replacing the λ, μ, ν by 1, 2, 3.

$$\frac{\sigma u}{\sigma_3 u}=\frac{1}{\sqrt{e_1-e_3}}\operatorname{sn}(\sqrt{e_1-e_3}.u,\ k)$$

$$\frac{\sigma_1 u}{\sigma_3 u}=\operatorname{cn}(\sqrt{e_1-e_3}.u,\ k)$$

$$\frac{\sigma_2 u}{\sigma_3 u}=\operatorname{dn}(\sqrt{e_1-e_3}.u,\ k).$$

Not all the sigma-quotients are so nearly identical with Jacobi's functions, but in all cases the argument u appears multiplied with the same factor $\sqrt{e_1-e_3}$ which is the largest of the three root-differences.

In the defining equation $k^2=\frac{e_2-e_3}{e_1-e_3}$ and the corresponding one $k'^2=\frac{e_1-e_2}{e_1-e_3}$ both of these quantities if real must be greater than zero and less than unity.

They will be real if the points in the plane representing e_1, e_2, e_3 lie in the same straight line, when mod. e_2 must be intermediate between mod. e_1 and mod. e_3 in magnitude. Then if we understand by K and K' the simplest values of the integrals

$$\int_0^1 \frac{dx}{\sqrt{1-x^2 . 1-k^2x^2}}; \quad \int_0^1 \frac{dx}{\sqrt{1-x^2 . 1-k'^2x^2}}$$

respectively, taking those values of the radicals whose real components are positive, we shall have

$$\omega_1\sqrt{e_1-e_3} = K, \quad \omega_3\sqrt{e_1-e_3} = iK',$$
$$\omega_2 = \omega_1 + \omega_3,$$

and $2\omega_1$, $2\omega_3$ are the primitive pair of periods for the before mentioned $\wp u$, so that as above

$$\wp\omega_1 = e_1, \quad \wp\omega_2 = e_2, \quad \wp\omega_3 = e_3.$$

It ought to be mentioned that $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$ can also be defined in the same simple manner as σu by means of infinite products. If we write

$$w_1 = (2\mu+1)\omega + 2\mu'\omega', \qquad w_2 = (2\mu+1)\omega + (2\mu'+1)\omega'$$
$$w_3 = 2\mu\omega + (2\mu'+1)\omega', \qquad [\mu, \mu' = 0, \pm 1, \pm 2 \ldots \pm \infty]$$

then in general, for $\lambda = 1, 2, 3$,

$$\sigma_\lambda u = e^{-\frac{1}{2}e_\lambda u^2} \Pi_{w_\lambda} \left(1 - \frac{u}{w_\lambda}\right) e^{\frac{u}{w_\lambda} + \frac{1}{2}\frac{u^2}{w_\lambda^2}}.$$

Finally to show the relation in which the sigma functions stand to the ϑ-functions of Jacobi, we find

$$\sigma u = \frac{2\omega}{\pi} e^{2\eta\omega v^2} \cdot \frac{2h^{\frac{1}{4}} \sin v\pi - 2h^{\frac{9}{4}} \sin 3v\pi + 2h^{\frac{25}{4}} \sin 5v\pi - \ldots}{2h^{\frac{1}{4}} - 3.2.h^{\frac{9}{4}} + 5.2h^{\frac{25}{4}} - \ldots} = 2\omega e^{2\eta\omega v^2} \cdot \frac{\vartheta_0(v)}{\vartheta_0'(o)}$$

$$\sigma_1 u = e^{2\eta\omega v^2} \cdot \frac{2h^{\frac{1}{4}} \cos v\pi + 2h^{\frac{9}{4}} \cos 3v\pi + 2h^{\frac{25}{4}} \cos 5v\pi + \ldots}{2h^{\frac{1}{4}} + 2h^{\frac{9}{4}} + 2h^{\frac{25}{4}} + \ldots} = e^{2\eta\omega v^2} \cdot \frac{\vartheta_1(v)}{\vartheta_1(o)}$$

$$\sigma_2 u = e^{2\eta\omega v^2} \cdot \frac{1 + 2h \cos 2v\pi + 2h^4 \cos 4v\pi + 2h^9 \cos 6v\pi + \ldots}{1 + 2h + 2h^4 + 2h^9 + \ldots} = e^{2\eta\omega v^2} \cdot \frac{\vartheta_2(v)}{\vartheta_2(o)}$$

$$\sigma_3 u = e^{2\eta\omega v^2} \cdot \frac{1 - 2h \cos 2v\pi + 2h^4 \cos 4v\pi - 2h^9 \cos 6v\pi + \ldots}{1 - 2h + 2h_4 - 2h^9 + \ldots} = e^{2\eta\omega v^2} \cdot \frac{\vartheta_3(v)}{\vartheta_3(o)}$$

where $\quad h = e^{\frac{\omega'}{\omega}\pi i}, \quad v = \dfrac{u}{2\omega}, \quad \eta = \dfrac{\sigma'\omega}{\sigma\omega}.$

The functions $\vartheta_0(v)$, $\vartheta_1(v)$, $\vartheta_2(v)$, $\vartheta_3(v)$ as here employed coincide respectively with Jacobi's $\vartheta_1(xq)$, $\vartheta_2(xq)$, $\vartheta_3(xq)$, $\vartheta(xq)$, if we write $v\pi = x$ and $h = q$.

But anything more than a slight account of Weierstrass' system, showing in particular its main points of contact with Jacobi's, would be beyond the intention of this paper. It is to be hoped that Weierstrass' ideas in the function-theory will soon find that widespread recognition which they undoubtedly merit. In a future paper I hope to exhibit the system in greater detail, in particular the formulæ of transformation, showing their analogies to the formulæ of Jacobi.

On Quadruple Theta-Functions.

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Part II.

In the following I employ the notation used by Schottky in his "*Abriss einer Theorie der Abelschen Functionen von drei Variabeln.*" On page 18, Schottky gives the fundamental theorem bearing upon his particular notation; it is as follows: "*Es ist möglich, ein System primitiver Indices*

$$1,\, 2,\, 3\, \ldots\, 2\rho + 1$$

und einen ausgezeichneten ε *so zu wählen, dass* εa *ein grader Index ist, wenn die Anzahl der primitiven Indices, aus denen* a *zusammengesetzt ist* $\equiv \rho$ *oder* $\rho + 1$ *mod.* 4 *ist, dagegen ein ungrader, wenn diese Anzahl* $\equiv \rho + 2$ *oder* $\rho - 1$ *mod.* 4 *ist.*"

For $\rho = 4$ the "primitive indices" are nine in number, and they may in general be denoted by the letters

$$k,\, l,\, m,\, n,\, p,\, q,\, r,\, s,\, t.$$

All of the characteristics of the quadruple theta-functions, with the exception of (0), may be represented by certain combinations of these letters, viz. by taking them one at a time, two at a time, three at a time and four at a time. We have thus:

	Number of cases.
The index (0)	1
The primitive indices taking one at a time	9
" " " two "	36
" " " three "	84
" " " four "	126
	256

The even functions, 136 in number, are given by the first, second and fifth of these cases, and the odd functions, 120 in number, by the third and fourth cases. That is, the even functions will have the suffixes o, k and $klmn$, and the odd functions will have the suffixes kl and klm. In the numbers of the *Annales de*

l'École Normale for June, July and August, 1883, M. Brunel has investigated the relations similar to the Göpel and Kummer relations for the double theta-functions which exist in the case of the triple theta-functions. I propose in what follows to employ Brunel's method in working out the corresponding relations connecting the quadruple functions. Brunel starts out from certain relations given by Schottky in the *Nachtrag* to the above mentioned book "*Ueber die hyperelliptischen Functionen dreier Variabeln,*" and uses a method which is fundamentally the same as that employed by Brioschi in his paper already referred to in Part I of this article,* but the manner in which he develops it is simpler than would be possible had he employed, without alteration, the method indicated by Brioschi.

I shall use almost without change the notation employed by Brunel, only altering it when the greater complexity of the present case makes it desirable. Following Schottky, write first:

$$\frac{L_k^4}{L_0^4}=\frac{-1}{(a_l-a_k)(a_m-a_k)(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)}$$

$$\frac{L_{kl}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_m-a_k)(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_m-a_l)(a_n-a_l)(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\end{array}\right.}$$

$$\frac{L_{klm}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_n-a_k)(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_n-a_l)(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\\ \times(a_n-a_m)(a_p-a_m)(a_q-a_m)(a_r-a_m)(a_s-a_m)(a_t-a_m)\end{array}\right.}$$

$$\frac{L_{klmn}^4}{L_0^4}=\frac{-1}{\left\{\begin{array}{l}(a_p-a_k)(a_q-a_k)(a_r-a_k)(a_s-a_k)(a_t-a_k)\\ \times(a_p-a_l)(a_q-a_l)(a_r-a_l)(a_s-a_l)(a_t-a_l)\\ \times(a_p-a_m)(a_q-a_m)(a_r-a_m)(a_s-a_m)(a_t-a_m)\\ \times(a_p-a_n)(a_q-a_n)(a_r-a_n)(a_s-a_n)(a_t-a_n)\end{array}\right.}$$

Now consider the functions P defined by the equations

1.
$$\frac{L_0}{L_0}P_0=\frac{\theta_0}{\theta_0},\quad \frac{L_k}{L_0}P_k=\frac{\theta_k}{\theta_0},$$
$$\frac{L_{kl}}{L_0}P_{kl}=\frac{\theta_{kl}}{\theta_0},\quad \frac{L_{klm}}{L_0}P_{klm}=\frac{\theta_{klm}}{\theta_0},$$
$$\frac{L_{klmn}}{L_0}P_{klmn}=\frac{\theta_{klmn}}{\theta_0},$$

* Brioschi.—*La relazione di Göpel per funzioni iperellittiche d'ordine qualunque.* Annali di Matematica, Serie II^a, Tomo X°.

then, following Weierstrass and Schottky, and writing

$$2.\quad \begin{aligned} R(x) &= (a_k - x)(a_l - x)(a_m - x)(a_n - x)(a_p - x)(a_q - x)(a_r - x)(a_s - x)(a_t - x) \\ \phi(x) &= (x - x_1)(x - x_2)(x - x_3)(x - x_4) \end{aligned}$$

we have

$$P_0 = 1 \quad \ldots \ldots \quad 1 \text{ function } P_0$$

$$P_k = \sqrt{(a_k - x_1)(a_k - x_2)(a_k - x_3)(a_k - x_4)} \quad \ldots \ldots \quad 9 \text{ functions } P_k$$

$$3.\quad P_{kl} = P_k P_l \sum_{i=1}^{i=4} \frac{\sqrt{R(x_i)}}{(a_k - x_i)(a_l - x_i)\,\varphi'(x_i)} \quad \ldots \ldots \quad 36 \text{ functions } P_{kl}$$

$$P_{klm} = P_k P_l P_m \sum_{i=1}^{i=4} \frac{\sqrt{R(x_i)}}{(a_k - x_i)(a_l - x_i)(a_m - x_i)\,\varphi'(x_i)} \quad \ldots \ldots \quad 84 \text{ functions } P_{klm}$$

$$P_{klmn} = P_k P_l P_m P_n \sum_{i=1}^{i=4} \frac{\sqrt{R(x_i)}}{(a_k - x_i)(a_l - x_i)(a_m - x_i)(a_n - x_i)\,\varphi'(x_i)} \ldots 126 \text{ functions } P_{klmn}$$

making in all 256 P-functions replacing the 256 Θ-functions. In these equations the letters k, l, m and n are all different from each other. We have now to determine the linear relations existing between the squares of these P-functions and those existing between their products taken two and two. Write

$$4.\quad \Sigma x_i = \alpha, \quad \Sigma x_i x_j = \beta, \quad \Sigma x_i x_j x_k = \gamma, \quad x_1 x_2 x_3 x_4 = \delta.$$

The summations to be taken from 1 to 4 and i, j, k all having different values. Further write

$$(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) = -\theta,$$

that is

$$5.\quad \begin{vmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -\theta;$$

and in general write

$$\begin{vmatrix} y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \\ y_1^{n-2} & y_2^{n-2} & \cdots & y_n^{n-2} \\ \cdot & \cdot & \cdots & \cdot \\ y_1 & y_2 & \cdots & y_n \\ 1 & 1 & \cdots & 1 \end{vmatrix} \equiv |y_1 y_2 \ldots y_n|,$$

that is

$$6.\quad |x_1 x_2 x_3 x_4| = -\theta.$$

The P-functions can now be written in the following manner,

$$P_0 = 1, \quad P_k = \sqrt{|a_k x_1||a_k x_2||a_k x_3||a_k x_4|}$$

$$P_{kl} = \frac{1}{|x_1 x_2 x_3 x_4|} \left\{ |x_2 x_3 x_4| \sqrt{R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4|}{|a_k x_1||a_l x_1|}} \right.$$
$$\left. - |x_3 x_4 x_1| \surd \quad + |x_4 x_1 x_2| \surd \quad - |x_1 x_2 x_3| \surd \right\}$$

7. $$P_{klm} = \frac{1}{|x_1 x_2 x_3 x_4|} \left\{ |x_2 x_3 x_4| \sqrt{R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4|}{|a_k x_1||a_l x_1||a_m x_1|}} \right.$$
$$\left. - |x_3 x_4 x_1| \surd \quad + |x_4 x_1 x_2| \surd \quad - |x_1 x_2 x_3| \surd \right\}$$

$$P_{klmn} =$$
$$\frac{1}{|x_1 x_2 x_3 x_4|} \left\{ |x_2 x_3 x_4| \sqrt{R(x_1) \frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4||a_n x_2||a_n x_3||a_n x_4|}{|a_k x_1||a_l x_1||a_m x_1||a_n x_1|}} \right.$$
$$\left. - |x_3 x_4 x_1| \surd \quad + |x_4 x_1 x_2| \surd \quad - |x_1 x_2 x_3| \surd \right\}$$

It is of course perfectly obvious how to fill up the empty radical signs. The above forms for the P-functions are retained for the same reason that Brunel gives in the case of the triple theta-functions, that is, although the denominators under the radical signs are actually factors of $R(x_1)$, $R(x_2)$, $R(x_3)$ and $R(x_4)$, it is more convenient in the following transformations to retain the fractional form.

Linear Relations between the Squares of the P-Functions.

Take first the case of the functions with a single index, *i. e.* P_k, P_l, etc. We have

8. $$P_k^2 = |a_k x_1||a_k x_2||a_k x_3||a_k x_4|;$$

expanding this and using the notation given above, we have

9. $$P_k^2 = a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta.$$

As this is linear in the quantities α, β, γ, δ, and as k is any one of the primitive indices, we can, by assuming any five such relations, eliminate α, β, γ and δ; the result of the elimination is obviously

10. $$\begin{vmatrix} P_k^2 - a_k^4, & P_l^2 - a_l^4, & P_m^2 - a_m^4, & P_n^2 - a_n^4, & P_p^2 - a_p^4 \\ a_k^3 & a_l^3 & a_m^3 & a_n^3 & a_p^3 \\ a_k^2 & a_l^2 & a_m^2 & a_n^2 & a_p^2 \\ a_k & a_l & a_m & a_n & a_p \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or expanding this we have

11. $$P_k^2 |a_l a_m a_n a_p| + P_l^2 |a_m a_n a_p a_k| + P_m^2 |a_n a_p a_k a_l|$$
$$+ P_n^2 |a_p a_k a_l a_m| + P_p^2 |a_k a_l a_m a_n| = |a_k a_l a_m a_n a_p| P_0^2$$

Since $P_0 = 1$, this factor may be introduced solely for the sake of symmetry. If instead of eliminating α, β, γ and δ between five relations of the form (9) we eliminate α, β, γ, δ and 1 between six such relations, we have obviously

$$12. \qquad \begin{vmatrix} P_k^2 & P_l^2 & P_m^2 & P_n^2 & P_p^2 & P_q^2 \\ a_k^4 & a_l^4 & a_m^4 & a_n^4 & a_p^4 & a_q^4 \\ a_k^3 & a_l^3 & a_m^3 & a_n^3 & a_p^3 & a_q^3 \\ a_k^2 & a_l^2 & a_m^2 & a_n^2 & a_p^2 & a_q^2 \\ a_k & a_l & a_m & a_n & a_p & a_q \\ 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0,$$

or expanding

$$13. \qquad P_k^2 \, | a_l a_m a_n a_p a_q | - P_l^2 \, | a_m a_n a_p a_q a_k | + P_m^2 \, | a_n a_p a_q a_k a_l | \\ - P_n^2 \, | a_p a_q a_k a_l a_m | + P_p^2 \, | a_q a_k a_l a_m a_n | - P_q^2 \, | a_k a_l a_m a_n a_p | = 0.$$

We have thus found the linear relations existing between the squares of the P-functions possessing a single suffix, or index, *i. e.* between the functions whose indices are

$$0 \; k \; l \; m \; n \; p \; q \; r \; s \; t,$$

and it is seen that these functions form a group of ten such that any five being given the square of any one of the remaining five can be expressed as a linear function of the squares of the chosen five. Following Brunel I shall call this the group 0.

Consider next the case of the P-functions with two suffixes: for the square of any one of them, say P_{kl}, we have

$$14. \qquad P_{kl}^2 = \frac{1}{|x_1x_2x_3x_4|^2} \Big\{ |x_2x_3x_4|^2 . R(x_1) \frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|}{|a_kx_1||a_lx_1|} + \ldots \\ - 2 |x_2x_3x_4||x_3x_4x_1| \sqrt{R(x_1)R(x_2)} \, |a_kx_3||a_kx_4||a_lx_3||a_lx_4| \\ + 2 |x_2x_3x_4||x_4x_1x_2| \sqrt{R(x_1)R(x_3)} \, |a_kx_4||a_kx_2||a_lx_4||a_lx_2| \\ - 2 |x_2x_3x_4||x_1x_2x_3| \sqrt{R(x_1)R(x_4)} \, |a_kx_2||a_kx_3||a_lx_2||a_lx_3| \\ - 2 |x_3x_4x_1||x_4x_1x_2| \sqrt{R(x_2)R(x_3)} \, |a_kx_1||a_kx_4||a_lx_1||a_lx_4| \\ + 2 |x_3x_4x_1||x_1x_2x_3| \sqrt{R(x_2)R(x_4)} \, |a_kx_1||a_kx_3||a_lx_1||a_lx_3| \\ - 2 |x_4x_1x_2||x_1x_2x_3| \sqrt{R(x_3)R(x_4)} \, |a_kx_1||a_kx_2||a_lx_1||a_lx_2| \Big\}.$$

It is possible to find a linear relation between four of these P-functions with two suffixes which is entirely rational, that is, a relation which shall not contain any of the quantities $\sqrt{R(x_i)R(x_j)}$. Take four of the functions P_{kl} which have the first suffix k in common, say P_{kl}, P_{km}, P_{kn}, P_{kp}, then in order that the radicals $\sqrt{R(x_i)R(x_j)}$ may disappear we must find a series of multipliers A, B, C, D, satisfying the equation

$$15. \qquad A|a_lx_3||a_lx_4| - B|a_mx_3||a_mx_4| + C|a_nx_3||a_nx_4| - D|a_px_3||a_px_4| = 0.$$

Giving A, B, C and D the following values,

16.
$$A=|a_m a_n a_p|,\quad B=|a_n a_p a_l|$$
$$C=|a_p a_l a_m|,\quad D=|a_l a_m a_n|$$

it is easy to see that equation 15 is satisfied. Assuming then four equations of the same form as 14, we have

17.
$$\begin{aligned}
&|a_m a_n a_p|\,P^2_{kl}-|a_n a_p a_l|\,P^2_{km}+|a_p a_k a_m|\,P^2_{kn}-|a_l a_m a_n|\,P^2_{kp}=\\
&\frac{1}{|x_1x_2x_3x_4|^2}\Big[\,|a_m a_n a_p|\,\Big\{\,|x_2x_3x_4|^2R(x_1)\,\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|}{|a_kx_1||a_lx_1|}+\ldots\Big\}\\
&\qquad-|a_n a_p a_l|\,\Big\{\,|x_2x_3x_4|^2R(x_1)\,\frac{|a_kx_2||a_kx_3||a_kx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_mx_1|}+\ldots\Big\}\\
&\qquad+|a_p a_l a_m|\,\Big\{\,|x_2x_3x_4|^2R(x_1)\,\frac{|a_kx_2||a_kx_3||a_kx_4||a_nx_2||a_nx_3||a_nx_4|}{|a_kx_1||a_nx_1|}+\ldots\Big\}\\
&\qquad-|a_l a_m a_n|\,\Big\{\,|x_2x_3x_4|^2R(x_1)\,\frac{|a_kx_2||a_kx_3||a_kx_4||a_px_2||a_px_3||a_px_4|}{|a_kx_1||a_px_1|}+\ldots\Big\}\Big].
\end{aligned}$$

Introducing the values of $R(x_1)$, $R(x_2)$, etc., it is not difficult to see that the first line of this equation may be thrown into the following form,

$$\begin{array}{l}
\dfrac{1}{|x_1x_2x_3x_4|^2}\{\,|a_ma_na_p||a_mx_1||a_nx_1||a_px_1|\,[\,|x_2x_3x_4||a_qx_1||a_rx_1||a_sx_1||a_tx_1||a_kx_2||a_kx_3||a_kx_4|\,]\,\mathfrak{A}\\
\qquad-|a_na_pa_l||a_nx_1||a_px_1||a_lx_1|\,[\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad]\,\mathfrak{B}\\
\qquad+|a_pa_la_m||a_px_1||a_lx_1||a_mx_1|\,[\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad]\,\mathfrak{C}\\
\qquad-|a_la_ma_n||a_lx_1||a_mx_1||a_nx_1|\,[\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad\text{“}\quad]\,\mathfrak{D}
\end{array}$$

where $\mathfrak{A}=|x_2x_3x_4||a_lx_2||a_lx_3||a_lx_4|-|x_3x_4x_1||a_lx_3||a_lx_4||a_lx_1|$
$\qquad+|x_4x_1x_2||a_lx_4||a_lx_1||a_lx_2|-|x_1x_2x_3||a_lx_1||a_lx_2||a_lx_3|$

and $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ are obtained by changing l into m, n and p respectively. We have then
$$\mathfrak{A}=\mathfrak{B}=\mathfrak{C}=\mathfrak{D}=|x_1x_2x_3x_4|=\text{ say }\Delta.$$
Write for convenience
$$|x_2x_3x_4||a_qx_1||a_rx_1||a_sx_1||a_tx_1||a_kx_2||a_kx_3||a_kx_4|=\Gamma_1,$$
then this becomes
$$\begin{aligned}\frac{\Delta\Gamma_1}{|x_1x_2x_3x_4|^2}\{\,&|a_ma_na_p||a_mx_1||a_nx_1||a_px_1|-|a_na_pa_l||a_nx_1||a_px_1||a_lx_1|\\
&+|a_pa_la_m||a_px_1||a_lx_1||a_mx_1|-|a_la_ma_n||a_lx_1||a_mx_1||a_nx_1|\,\}.\end{aligned}$$

The term in the $\{\}$ is easily seen to be equal to $|a_la_ma_na_p|$. Equation 17 thus takes the form,

18.
$$\begin{aligned}&|a_ma_na_p|\,P^2_{kl}-|a_na_pa_l|\,P^2_{km}+|a_pa_la_m|\,P^2_{kn}-|a_la_ma_n|\,P^2_{kp}=\\
&\qquad\frac{|a_la_ma_na_p|}{|x_1x_2x_3x_4|}\,[\Gamma_1-\Gamma_2+\Gamma_3-\Gamma_4].\end{aligned}$$

Remembering to pay attention to the signs, we may write the term in [ ] as $\Sigma\Gamma$; then writing

19. $$\Sigma a_q = \lambda,\ \Sigma a_q a_r = \mu,\ \Sigma a_q a_r a_s = \nu,\ \Sigma a_q a_r a_s a_t = \pi,$$

we have

20. $$\Sigma\Gamma = \Sigma\,|x_2x_3x_4|\,(\pi - x_1\nu + x_1^2\mu - x_1^3\lambda + x_1^4)\big(a_k^3 - a_k^2(x_2+x_3+x_4) + a_k(x_2x_3 + x_4x_2 + x_3x_4) - x_2x_3x_4\big).$$

Now writing as above $\theta = -|x_1x_2x_3x_4|$

and introducing the abbreviations α, β, γ, δ and λ, μ, ν, π, it is not difficult to see that

21. $$\theta = |x_1x_2x_3x_4|\,\{a_k^4 - a_k^3\alpha + a_k^2\beta - a_k\gamma + \delta - a_k^4 + a_k^3\lambda - a_k^2\mu + a_k\nu - \pi\};$$

referring now to equation 9 we have

22. $$\theta = |x_1x_2x_3x_4|\,\{P_k^2 - |a_ka_p||a_ka_q||a_ka_r||a_ka_s||a_ka_t|\,P_0^2\}.$$

Expanding equation 18 it becomes

$$\begin{aligned}
&a_k^3\{\pi\Sigma\,|x_1x_2x_3| - \nu\Sigma x_1\,|x_2x_3x_4| + \mu\Sigma x_1^2\,|x_2x_3x_4| - \lambda\Sigma x_1^3\,|x_2x_3x_4| + \Sigma x_1^4\,|x_2x_3x_4|\,\}\\
&- a_k^2\{\pi\Sigma(x_2+x_3+x_4)\,|x_2x_3x_4| - \nu\Sigma x_1(x_2+x_3+x_4)\,|x_2x_3x_4| + \mu\Sigma x_1^2(x_2+x_3+x_4)\,|x_2x_3x_4|\\
&\quad - \lambda\Sigma x_1^3(x_2+x_3+x_4)\,|x_2x_3x_4| + \Sigma x_1^4(x_2+x_3+x_4)\,|x_2x_3x_4|\,\}\\
&\quad + a_k\{\pi\Sigma(x_2x_3+x_4x_2+x_3x_4)\,|x_2x_3x_4| - \nu\Sigma x_1(x_2x_3+x_4x_2+x_3x_4)\,|x_2x_3x_4|\\
&\quad + \mu\Sigma x_1^2(x_2x_3+x_4x_2+x_3x_4)\,|x_2x_3x_4| - \lambda\Sigma x_1^3(x_2x_3+x_4x_2+x_3x_4)\,|x_2x_3x_4|\\
&\quad + \Sigma x_1^4(x_2x_3+x_4x_2+x_3x_4)\,|x_2x_3x_4|\,\} - \{\pi\Sigma\, x_2x_3x_4\,|x_2x_3x_4|\\
&\quad - \nu\, x_1x_2x_3x_4\Sigma\,|x_2x_3x_4| + \mu\, x_1x_2x_3x_4\Sigma x_1\,|x_2x_3x_4| - \lambda\, x_1x_2x_3x_4\Sigma x_1^2\,|x_2x_3x_4|\\
&\quad + x_1x_2x_3x_4\Sigma x_1^3\,|x_2x_3x_4|\,\} =\\
&\qquad |a_ma_na_p|\,P_{kl}^2 - |a_na_pa_l|\,P_{km}^2 + |a_pa_la_m|\,P_{kn}^2 - |a_la_ma_n|\,P_{kp}^2.
\end{aligned}$$

Of course in all these summations particular care must be taken to give the right signs to each term; for example, $\Sigma\,|x_2x_3x_4|$ means

$$|x_2x_3x_4| - |x_3x_4x_1| + |x_4x_1x_2| - |x_1x_2x_3|.$$

Using now equations 19 to 21 inclusive, we have, after simple reductions, for the reduced form of equation 18,

23. $$\begin{aligned}&|a_ma_na_p|\,P_{kl}^2 - |a_na_pa_l|\,P_{km}^2 + |a_pa_la_m|\,P_{kn}^2 - |a_la_ma_n|\,P_{kp}^2 =\\ &|a_la_ma_na_p|\,\{P_k^2 - |a_ka_p||a_ka_q||a_ka_r||a_ka_s||a_ka_t|\,P_0^2\}.\end{aligned}$$

The factor $P_0^2 = 1$ being introduced simply for the sake of symmetry.

Now advance all the letters after k, that is, change l, m, n, p, q, r, s, t into m, n, p, q, r, s, t, l, and 23 becomes

24. $$\begin{aligned}&|a_na_pa_q|\,P_{km}^2 - |a_pa_qa_m|\,P_{kn}^2 + |a_qa_ma_n|\,P_{kp}^2 - |a_ma_na_p|\,P_{kq}^2 =\\ &|a_ma_na_pa_q|\,\{P_k^2 - |a_ka_q||a_ka_r||a_ka_s||a_ka_t||a_ka_l|\,P_0^2\}.\end{aligned}$$

The coefficients of P_0^2 in 23 and 24 are respectively

$$[(a_m-a_n)(a_m-a_p)(a_n-a_p)(a_k-a_q)(a_k-a_r)(a_k-a_s)(a_k-a_t)]$$
$$(a_l-a_m)(a_l-a_n)(a_l-a_p)(a_k-a_p)$$

and

$$[(a_m-a_n)(a_m-a_p)(a_n-a_p)(a_k-a_q)(a_k-a_r)(a_k-a_s)(a_k-a_t)]$$
$$(a_m-a_q)(a_n-a_q)(a_p-a_q)(a_k-a_l).$$

Multiplying then 23 by the factor

$$(a_m-a_q)(a_n-a_q)(a_p-a_q)(a_k-a_l),$$

and 24 by the factor $(a_l-a_m)(a_l-a_n)(a_l-a_p)(a_k-a_p)$,

and subtracting one result from the other we eliminate P_0 and have

$$\begin{aligned}
&P_{kl}^2\,|a_ma_na_pa_q||a_ka_l| \\
&-P_{km}^2\{\,|a_na_pa_l||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_na_pa_q||a_la_m||a_la_n||a_la_p||a_ka_p|\,\} \\
&+P_{kn}^2\{\,|a_pa_la_m||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_pa_qa_m||a_la_m||a_la_n||a_la_p||a_ka_p|\,\} \\
&-P_{kp}^2\{\,|a_la_ma_n||a_ma_q||a_na_q||a_pa_q||a_ka_l|+|a_qa'_ma_n||a_la_m||a_la_n||a_la_p||a_ka_p|\,\} \\
&+P_{kq}^2\{\,|a_ma_na_p||a_la_m||a_la_n||a_la_p||a_ka_p|\,\}= \\
&P_k^2\{a_la_ma_na_p||a_ma_q||a_na_q||a_pa_q||a_ka_l|-|a_ma_na_pa_q||a_la_m||a_la_n||a_la_p||a_ka_p|\,\}
\end{aligned}$$

This reduces to

$$\begin{aligned}
&P_{kl}^2\,|a_ma_na_pa_q||a_ka_l|-P_{km}^2\,|a_na_pa_qa_l||a_ka_m|+P_{kn}^2\,|a_pa_qa_la_m||a_ka_n| \\
&-P_{kp}^2\,|a_qa_la_ma_n||a_ka_p|+P_{kq}^2\,|a_la_ma_na_p||a_ka_q|= \\
&P_k^2\,|a_la_ma_na_pa_q|.
\end{aligned} \tag{25.}$$

If we here make again the substitution

$$\left|\begin{matrix} l & m & n & p & q & r & s & t \\ m & n & p & q & r & s & t & l \end{matrix}\right|$$

we get a new relation connecting the squares of P_k, P_{km}, P_{kn}, P_{kp}, P_{kq}, P_{kr}. The coefficient of P_k^2 will be $|a_ma_na_pa_qa_r|$:

multiplying this equation then by the factor

$$|a_la_m||a_la_n||a_la_p||a_la_q|,$$

and multiplying 25 by $|a_ma_r||a_na_r||a_pa_r||a_qa_r|$,

and subtracting one result from the other, we eliminate P_k and have a linear relation between the squares of

$$P_{kl},\ P_{km},\ P_{kn},\ P_{kp},\ P_{kq},\ P_{kr},\ \text{viz.}$$

$$\begin{aligned}
&P_{kl}^2\,|a_ma_na_pa_qa_r||a_ka_l|-P_{km}^2\,|a_na_pa_qa_ra_l||a_ka_m| \\
&+P_{kn}^2\,|a_pa_qa_ra_la_m||a_ka_n|-P_{kp}\,|a_qa_ra_la_ma_n||a_ka_p| \\
&+P_{kq}^2\,|a_ra_la_ma_na_p||a_ka_q|-P_{kr}\,|a_la_ma_na_pa_q||a_ka_r|.
\end{aligned} \tag{26.}$$

It is obvious that we might have eliminated P_k^2 between 23 and 24, and so have found a linear relation connecting the squares of

$$P_0,\ P_{kl},\ P_{km},\ P_{kn},\ P_{kp},\ P_{kq},$$

and by making the above substitution and eliminating P_0^2 between the two equations thus formed we would again arrive at 26. It is then clear that the functions with the indices

$$0,\ k,\ kl,\ km,\ kn,\ kp,\ kq,\ kr,\ ks,\ kt,$$

form a group of ten, such that any five being selected the squares of any of the remaining five can be expressed as a linear function of the squares of the chosen five. There are of course in all nine such groups, and these may be tabulated as follows:

0	k	kl	km	kn	kp	kq	kr	ks	kt
0	l	lk	lm	ln	lp	lq	lr	ls	lt
0	m	mk	ml	mn	mp	mq	mr	ms	mt
0	n	nk	nl	nm	np	nq	nr	ns	nt
0	p	pk	pl	pm	pn	pq	pr	ps	pt
0	q	qk	ql	qm	qn	qp	qr	qs	qt
0	r	rk	rl	rm	rn	rp	rq	rs	rt
0	s	sk	sl	sm	sn	sp	sq	sr	st
0	t	tk	tl	tm	tn	tp	tq	tr	ts

and the groups will be called the k-group, the l-group, etc.

We will now take up the case of three indices, which, as will be seen, divides into two sub-cases, according to the choice of the index. The two sub-cases give rise to two tables, the first containing 36 groups and the second containing 84 groups. As the method of working out these groups by Brunel's method has already been sufficiently indicated, I shall, in what follows, leave out as much as possible the purely algebraical processes of reduction, as they now become very long and wholly uninteresting. Squaring the function P_{klm} we have

$$27.\quad P_{klm}^2 = \frac{1}{|x_1x_2x_3x_4|^2}\Big\{ |x_2x_3x_4|^2 R(x_1) \frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_lx_1||a_mx_1|} + \ldots$$
$$- 2|x_2x_3x_4|\sqrt{R(x_1)R(x_2)}\,|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_mx_3||a_mx_4| + \ldots - \ldots + \ldots \Big\}$$

The radicals $\sqrt{R(x_1)R(x_2)}$, etc., may be eliminated between any six equations of the form 27, or between five equations of this form, having each a common index, say k, or between four equations having each two indices, say k and l, common. Choose multipliers A, B, C and D, such that the coefficient of $\sqrt{R(x_1)R(x_2)}$ (and in consequence the coefficients of all the other radicals) shall be zero in the sum $AP_{klm}^2 + BP_{kln}^2 + CP_{klp}^2 + DP_{kl}^2$.

This coefficient is easily seen to be

$$A|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_mx_3||a_mx_4| + B|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_nx_3||a_nx_4|$$
$$+ C|a_kx_3||a_kx_4||a_lx_3||a_lx_4||a_px_3||a_px_4| + D|a_kx_3||a_kx_4||a_lx_3||a_lx_4|$$

Striking out the common factor

$$|a_k x_3||a_k x_4||a_l x_3||a_l x_4|$$

the condition to be satisfied is

28. $$A|a_m x_3||a_m x_4| + B|a_n x_3||a_n x_4| + C|a_p x_3||a_p x_4| + D = 0\,;$$

this is equivalent to

$$A + B + C = 0,$$
$$a_m A + a_n B + a_p C = 0,$$
$$a_m^2 A + a_n^2 B + a_p^2 C + D = 0.$$

These are easily seen to be satisfied by the values

29. $$A = |a_n a_p|,\ \ B = |a_p a_m|,\ \ C = |a_m a_n|,\ \ D = -|a_m a_n a_p|.$$

Introducing then these values of A, B, C and D, we have

$$|a_n a_p|P_{klm}^2 + |a_p a_m|P_{kln}^2 + |a_m a_n|P_{klp}^2 - |a_m a_n a_p|P_{kl}^2 =$$

$$\frac{1}{|x_1x_2x_3x_4|^2}\Big\{|a_n a_p|\Big[|x_2x_3x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_m x_2||a_m x_3||a_m x_4|}{|a_k x_1||a_l x_1||a_m x_1|} + \ldots\Big]$$

$$+|a_p a_m|\Big[|x_2x_3x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_n x_2||a_n x_3||a_n x_4|}{|a_k x_1||a_l x_1||a_n x_1|} + \ldots\Big]$$

30. $$+|a_m a_n|\Big[|x_2x_3x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4||a_p x_2||a_p x_3||a_p x_4|}{|a_k x_1||a_l x_1||a_p x_1|} + \ldots\Big]$$

$$-|a_m a_n a_p|\Big[|x_2x_3x_4|^2 R(x_1)\frac{|a_k x_2||a_k x_3||a_k x_4||a_l x_2||a_l x_3||a_l x_4|}{|a_k x_1||a_l x_1|} + \ldots\Big]\Big\}$$

This is to be reduced just as in the case of two indices, viz. expand 30, so that the first line becomes

$$\{|a_n a_p||x_2x_3x_4||a_n x_1||a_p x_1|\ldots|a_l x_1||a_k x_2||a_k x_3||a_k x_4||a_l x_2|\ldots|a_l x_4|\}$$

31. $$\times\{|x_2x_3x_4||a_m x_2||a_m x_3||a_m x_4| - |x_3x_4x_1||a_m x_3||a_m x_4||a_m x_1| + \ldots$$
$$- |x_1x_2x_3||a_m x_1||a_m x_2||a_m x_3|\} + \ldots$$

There are three more terms similar to this to be obtained by simply advancing certain of the subscripts. The remaining three lines in 30 are to be expanded in a similar manner, and then the terms which have been introduced will disappear by aid of equations 28 and 29. The right-hand side of 30 is now easily reduced by aid of the following identities:

$$|x_2x_3x_4| - |x_3x_4x_1| + |x_4x_1x_2| - |x_1x_2x_3| = 0,$$

$$(x_2 + x_3 + x_4)|x_2x_3x_4| - (x_3 + x_4 + x_1)|x_3x_4x_1|$$

32. $$+ (x_4 + x_1 + x_2)|x_4x_1x_2| - (x_1 + x_2 + x_3)|x_1x_2x_3| = 0,$$

$$(x_2x_3 + x_3x_4 + x_4x_2)|x_2x_3x_4| - (x_3x_4 + x_4x_1 + x_1x_3)|x_3x_4x_1|$$
$$+ (x_4x_1 + x_1x_2 + x_2x_4)|x_4x_1x_2| - (x_1x_2 + x_2x_3 + x_3x_1)|x_1x_2x_3| = 0,$$

$$x_2x_3x_4|x_2x_3x_4| - x_3x_4x_1|x_3x_4x_1| + x_4x_1x_2|x_4x_1x_2| - x_1x_2x_3|x_1x_2x_3| = -|x_1x_2x_3x_4|.$$

This last equation written out in full is

33. $$\begin{vmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} x_2x_3x_4 & x_3x_4x_1 & x_4x_1x_2 & x_1x_2x_3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

which is clearly true, as each of the determinants is equal to the product of differences of the x's.

Another form of expressing $|x_1x_2x_3x_4|$ is obviously

34. $$\begin{vmatrix} x_1^3 & x_2^3 & x_3^3 & x_4^3 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = - x_1^3x_2^3x_3^3x_4^3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{x_1} & \frac{1}{x_2} & \frac{1}{x_3} & \frac{1}{x_4} \\ \frac{1}{x_1^2} & \frac{1}{x_2^2} & \frac{1}{x_3^2} & \frac{1}{x_4^2} \\ \frac{1}{x_1^3} & \frac{1}{x_2^3} & \frac{1}{x_3^3} & \frac{1}{x_4^3} \end{vmatrix}.$$

For the general case of n variables $x_1, x_2 \ldots x_n$, it is not difficult to see that $|x_1, x_2 \ldots x_n|$ satisfies the identity

35. $$\begin{vmatrix} x_1^{n-1}, & x_2^{n-1} \ldots & x_n^{n-1} \\ x_1^{n-2}, & x_2^{n-2} \ldots & x_n^{n-2} \\ \ldots & \ldots & \ldots \\ x_1, & x_2, \ldots & x_n \\ 1 & 1 \ldots & 1 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} x_2x_3 \ldots x_n, & x_3x_4 \ldots x_nx_1 \ldots, & x_1x_2 \ldots x_{n-1} \\ x_1^{n-2}, & x_2^{n-2} \ldots\ldots, & x_n^{n-2} \\ x_1^{n-3}, & x_2^{n-3} \ldots\ldots, & x_n^{n-3} \\ \ldots & \ldots & \ldots \\ x_1, & x_2 \ldots\ldots & x_n \\ 1 & 1 \ldots\ldots & 1 \end{vmatrix}$$

or

36. $$\begin{vmatrix} x_1^{n-1}, & x_2^{n-1} \ldots & x_n^{n-1} \\ x_1^{n-2}, & x_2^{n-2} \ldots & x_n^{n-2} \\ \ldots & \ldots & \ldots \\ x_1 & x_2 \ldots & x_n \\ 1 & 1 \ldots & 1 \end{vmatrix} = (-)^{n-1} [x_1x_2 \ldots x_n]^{n-1} \begin{vmatrix} 1 & 1 \ldots & 1 \\ \frac{1}{x_1} & \frac{1}{x_2} \ldots & \frac{1}{x_n} \\ \frac{1}{x_1^2} & \frac{1}{x_2^2} \ldots & \frac{1}{x_n^2} \\ \ldots & \ldots & \ldots \\ \frac{1}{x_1^{n-1}} & \frac{1}{x_2^{n-1}} \ldots & \frac{1}{x_n^{n-1}} \end{vmatrix}.$$

The following identities are also very important in the sequence, viz. writing

37. $A = |a_na_pa_q|$, $B = -|a_pa_qa_m|$, $C = |a_qa_ma_n|$, $D = -|a_ma_na_p|$,

we have $$A + B + C + D = 0,$$

38. $$a_mA + a_nB + a_pC + a_qD = 0,$$
$$a_m^2A + a_n^2B + a_p^2C + a_q^2D = 0.$$

A similar group of identities may easily be written down for the general case. Using these last identities and taking the four functions

$$P^2_{klm},\ P^2_{kln},\ P^2_{klp},\ P^2_{klq},$$

we can, by multiplying the first by A, the second by B, etc., and taking the sum, eliminate the radicals $\sqrt{R(x_1)R(x_2)}$, $\sqrt{R(x_1)R(x_3)}$, etc. Forming this sum it is only necessary to show that

$$39. \qquad A\,|a_m x_3||a_m x_4| + B\,|a_n x_3||a_n x_4| + C\,|a_p x_3||a_p x_4| + D\,|a_q x_3||a_q x_4| = 0;$$

and this equation of condition is at once seen to be satisfied by the values

$$39'. \qquad A = |a_n a_p a_q|,\quad B = -|a_p a_q a_m|,\quad C = |a_q a_m a_n|,\quad D = -|a_m a_n a_p|.$$

We have now

$$\begin{aligned}
&|a_n a_p a_q|\,P^2_{klm} - |a_p a_q a_m|\,P^2_{kln} + |a_q a_m a_n|\,P^2_{klp} - |a_m a_n a_p|\,P^2_{klq} = \\
&\frac{1}{|x_1x_2x_3x_4|^2}\Big\{|a_n a_p a_q|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_mx_2||a_mx_3||a_mx_4|}{|a_kx_1||a_lx_1||a_mx_1|}+\ldots\Big] \\
40.\qquad &-|a_p a_q a_m|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_nx_2||a_nx_3||a_nx_4|}{|a_kx_1||a_lx_1||a_nx_1|}+\ldots\Big] \\
&+|a_q a_m a_n|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_px_2||a_px_3||a_px_4|}{|a_kx_1||a_lx_1||a_px_1|}+\ldots\Big] \\
&-|a_m a_n a_p|\Big[|x_2x_3x_4|^2R(x_1)\frac{|a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4||a_qx_2||a_qx_3||a_qx_4|}{|a_kx_1||a_lx_1||a_qx_1|}+\ldots\Big]\Big\}.
\end{aligned}$$

This may be briefly written in the form

$$\begin{aligned}
&|a_n a_p a_q|\,P^2_{klm} - |a_p a_q a_m|\,P^2_{kln} + |a_q a_m a_n|\,P^2_{klp} - |a_m a_n a_p|\,P^2_{klq} = \\
&\frac{1}{|x_1x_2x_3x_4|^2}\Big\{\ |a_n a_p a_q|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,\,a_l,\,a_m][x_2,\,x_3,\,x_4]}{[a_k,\,a_l,\,a_m][x_1]}+\ldots\Big] \\
41.\qquad &-|a_p a_q a_m|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,\,a_l,\,a_n][x_2,\,x_3,\,x_4]}{[a_k,\,a_l,\,a_n][x_1]}+\ldots\Big] \\
&+|a_q a_m a_n|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,\,a_l,\,a_p][x_2,\,x_3,\,x_4]}{[a_k,\,a_l,\,a_p][x_1]}+\ldots\Big] \\
&-|a_m a_n a_p|\Big[|x_2x_3x_4|^2R(x_1)\frac{[a_k,\,a_l,\,a_q][x_2,\,x_3,\,x_4]}{[a_k,\,a_l,\,a_q][x_1]}+\ldots\Big]\Big\}
\end{aligned}$$

Expanding this just as in the case of two indices and the case of equation 30, we have for the first line on the right-hand side of the equation

$$\begin{aligned}
&\{\,|a_n a_p a_q||x_2x_3x_4|\,[a_n,\,a_p,\,a_q,\,a_r,\,a_s,\,a_t][x_1][a_k,\,a_l][x_2,\,x_3,\,x_4]\,\} \\
&\times\{\,|x_2x_3x_4|\,[a_m][x_2,\,x_3,\,x_4| - |x_3x_4x_1|\,[a_m][x_3,\,x_4,\,x_1] \\
&\qquad + |x_4x_1x_2|\,[a_m][x_4,\,x_1,\,x_2] - |x_1x_2x_3|\,[a_m][x_1,\,x_2,\,x_3]\,\}
\end{aligned}$$

+ three similar terms.

The remaining three lines of 41 are to be expanded in the same manner, and then it will at once be seen that the extra terms which have been introduced

will vanish on account of the relations 39 and 39′. Consider now the terms containing the factor $R(x_1)$: they have obviously the common factor

$$|x_2x_3x_4|[a_r, a_s, a_t][x_1].[a_k, a_l][x_2, x_3, x_4]$$

and the remaining factor is

$$\begin{aligned}&|a_na_pa_q|[a_n, a_p, a_q][x_1]-|a_pa_qa_m|[a_p, a_q, a_m][x_1]\\&+|a_qa_ma_n|[a_q, a_m, a_n][x_1]-|a_ma_na_p|[a_m, a_n, a_p][x_1]=,\text{ say } K.\end{aligned}$$

Expanding K and using the identities 32 and 33, we have

42. $$K=-|a_ma_na_pa_q|.$$

The first line on the right-hand side of equations 40 or 41 contains four terms, the first of which contains the factor already mentioned, viz.

43. $$\begin{aligned}&|x_2x_3x_4|[a_m][x_2, x_3, x_4]-|x_3x_4x_1|[a_m][x_3, x_4, x_1]+|x_4x_1x_2|[a_m][x_4, x_1, x_2]\\&\qquad-|x_1x_2x_3|[a_m][x_1, x_2, x_3];\end{aligned}$$

the factor in each of the remaining terms is derived from this by changing m into n, p, q respectively. The same is true for the remaining three lines on the right-hand side of 40 or 41. This factor is independent of a_m, and the others not written down are equally independent of n, p, or q; for writing 43 out in full it is

$$\begin{aligned}&|x_2x_3x_4||a_mx_2||a_mx_3||a_mx_4|-|x_3x_4x_1||a_mx_3||a_mx_4||a_mx_1|\\&+|x_4x_1x_2||a_mx_4||a_mx_1||a_mx_2|-|x_1x_2x_3||a_mx_1||a_mx_2||a_mx_3|\end{aligned}$$

and this is equal to

$$\begin{aligned}&a_m^3\Sigma|x_2x_3x_4|+a_m^2\Sigma(x_2+x_3+x_4)|x_2x_3x_4|\\&\qquad+a_m\Sigma(x_2x_3+x_3x_4+x_4x_2)|x_2x_3x_4|+\Sigma x_2x_3x_4|x_2x_3x_4|.\end{aligned}$$

The first three terms of this vanish by virtue of the identities 32, and the fourth term by 33 becomes $=-|x_1x_2x_3x_4|$.

The right-hand side of 40 and 41 thus contains the factor

$$-|a_ma_na_pa_q|.-|x_1x_2x_3x_4|=|a_ma_na_pa_q||x_1x_2x_3x_4|.$$

The right-hand member of 41 takes now the form

$$\frac{1}{|x_1x_2x_3x_4|}|a_ma_na_pa_q|\Sigma|x_2x_3x_4|[a_r, a_s, a_t][x_1][a_k, a_l][x_2, x_3, x_4].$$

The Σ of course refers only to the cyclic permutations of the suffixes 1, 2, 3, 4. We have now to determine the value of the quantity under the summation sign, viz.

$$\Sigma|x_2x_3x_4||a_rx_1||a_sx_1||a_tx_1||a_kx_2||a_kx_3||a_kx_4||a_lx_2||a_lx_3||a_lx_4|$$

in order to find the relation connecting the squares of P_{klm}, P_{kln}, P_{klp}, P_{klq}. For greater generality and completeness, however, it is better to go back to equation 30 and reduce it, *i. e.* find the linear relation connecting the squares of P_{klm}, P_{kln}, P_{klp}, P_{kl}. It will then be seen that by making the substitution

$$\begin{vmatrix}k & l & m & n & p & q & r & s & t\\ l & m & n & p & q & r & s & t & k\end{vmatrix}$$

and eliminating one quantity, we arrive at what we might obtain directly by completing the reduction of equation 40.

The second factor in 31 is easily seen to be

44. $$= - | x_1 x_2 x_3 x_4 | ,$$

and the same is true for the corresponding factors in the remaining three lines of 30. Now adding together the first factors of the four lines in 30, viz. those similar to the first factor in 31, we have

$$\begin{aligned} & |a_n a_p| |x_2 x_3 x_4| [a_n, a_p \ldots a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ + & |a_p a_m| |x_2 x_3 x_4| [a_p, a_q \ldots a_t, a_m][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ + & |a_m a_n| |x_2 x_3 x_4| [a_m, a_p \ldots a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] . \end{aligned}$$

The fourth line need not be written down, as its second factor is zero.

Adding these terms we have

$$\begin{aligned} & \Sigma \{ |x_2 x_3 x_4| [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ & \times |a_n a_p| |a_n x_1| |a_p x_1| + |a_p a_m| |a_p x_1| |a_m x_1| + |a_m a_n| |a_m x_1| |a_n x_1| \} . \end{aligned}$$

Now $$|a_n a_p| + |a_p a_m| + |a_m a_n| = 0 ,$$

$$(a_n + a_p) |a_n a_p| + (a_p + a_m) |a_p a_n| + (a_m + a_n) |a_m a_n| = 0 ,$$

and $$a_n a_p |a_n a_p| + a_p a_m |a_p a_m| + a_m a_n |a_m a_n| = |a_m a_n a_p| ,$$

so that the above reduces to

45. $$|a_m a_n a_p| \Sigma |x_2 x_3 x_4| [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] ,$$

the summation referring to the subscripts 1, 2, 3, 4. Equation 30, or equation 31, becomes now, by taking into acccount 44 and 45,

46. $$\begin{aligned} & |a_n a_p| P^2_{klm} + |a_p a_m| P^2_{kln} + |a_m a_n| P^2_{klp} - |a_m a_n a_p| P^2_{kl} = \\ & \frac{1}{|x_1 x_2 x_3 x_4|} |a_m a_n a_p| \Sigma |x_2 x_3 x_4| [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] . \end{aligned}$$

For brevity write as before

$$\Sigma a_q = \lambda, \ \Sigma a_q a_r = \mu, \ \Sigma a_q a_r a_s = \nu, \ a_q a_r a_s a_t = \pi,$$

the summations extending over the subscripts q, r, s, t. We have now

47. $$\begin{aligned} & \Sigma |x_2 x_3 x_4| [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ = & \Sigma |x_2 x_3 x_4| (\pi - x_1 \nu + x_1^2 \mu - x_1^3 \lambda + x_1^4) \{ (a_k^3 - a_k^2 (x_2 + x_3 + x_4) + a_k (x_2 x_3 + x_3 x_4 + x_4 x_2) - x_2 x_3 x_4) \\ & \quad \times (a_l^3 - a_l^2 (x_2 + x_3 + x_4) + a_l (x_2 x_3 + x_3 x_4 + x_4 x_2) - x_2 x_3 x_4) \} \\ = & \Sigma |x_2 x_3 x_4| (\pi - x_1 \nu + x_1^2 \mu - x_1^3 \lambda + x_1^4) \{ a_k^3 a_l^3 - a_k^2 a_l^2 (a_k + a_l)(x_2 + x_3 + x_4) \\ & + a_k^2 a_l^2 (x_2 + x_3 + x_4)^2 + a_k a_l (a_k^2 + a_l^2)(x_2 x_3 + x_3 x_4 + x_4 x_2) \\ & - a_k a_l (a_k + a_l)(x_2 + x_3 + x_4)(x_2 x_3 + x_3 x_4 + x_4 x_2) + a_k a_l (x_2 x_3 + x_3 x_4 + x_4 x_2)^2 \\ & + (a_k^2 + a_l^2)(x_2 + x_3 + x_4) x_2 x_3 x_4 - (a_k + a_l)(x_2 x_3 + x_3 x_4 + x_4 x_2) x_2 x_3 x_4 \\ & - (a_k^3 + a_l^3) x_2 x_3 x_4 + x_2^2 x_3^2 x_4^2 \} . \end{aligned}$$

The identities given above and a number of similar ones serve for the reduction of this. Paying attention to the signs we have, for a few of the identities, the following:

$$\Sigma x_1^3 \,|\, x_2x_3x_4 \,| = -\,\theta$$
$$\Sigma (x_2^2 + x_3^2 + x_4^2) \,|\, x_2x_3x_4 \,| = +\,\theta$$
$$\Sigma x_2x_3x_4 \,|\, x_2x_3x_4 \,| = -\,\theta$$
$$\Sigma x_1 (x_2x_3 + x_3x_4 + x_4x_2) \,|\, x_2x_3x_4 \,| = +\,\theta$$
$$\Sigma x_1^4 \,|\, x_2x_3x_4 \,| = -\,\theta\alpha$$
$$\Sigma x_1^3 (x_2 + x_3 + x_4) \,|\, x_2x_3x_4 \,| = 0$$
$$\Sigma x_1^2 (x_2^2 + x_3^2 + x_4^2) \,|\, x_2x_3x_4 \,| = \theta\alpha, \text{ etc., etc., etc.}$$

It is easy to write down any number of these.

In the expansion of 47 the term containing the factor π is, on multiplying both sides of the equation by $|a_k,\, a_l|$

$$\pi \,|\, a_k a_l \,| \,\{ a_k^3 a_l^3 \Sigma \,|\, x_2x_3x_4 \,| - a_k^2 a_l^2 (a_k + a_l) \Sigma (x_2 + x_3 + x_4) \,|\, x_2x_3x_4 \,|$$
$$- a_k^2 a_l^2 \Sigma (x_2 + x_3 + x_4)^2 \,|\, x_2x_3x_4 \,|$$
$$+ a_k a_l (a_k^2 + a_l^2) \Sigma (x_2x_3 + x_3x_4 + x_4x_2) \,|\, x_2x_3x_4 \,|$$
$$- a_k a_l (a_k + a_l) \Sigma (x_2 + x_3 + x_4)(x_2x_3 + x_3x_4 + x_4x_2) \,|\, x_2x_3x_4 \,|$$
$$- a_k a_l \Sigma (x_2x_3 + x_3x_4 + x_4x_2)^2 \,|\, x_2x_3x_4 \,|$$
$$-(a_k^2 + a_l^2) \Sigma x_2x_3x_4 (x_2 + x_3 + x_4) \,|\, x_2x_3x_4 \,| - (a_k + a_l) \Sigma x_2x_3x_4 (x_2x_3 + x_3x_4 + x_4x_2) \,|\, x_2x_3x_4 \,|$$
$$-(a_k^3 + a_l^3) \Sigma x_2x_3x_4 \,|\, x_2x_3x_4 \,| + \Sigma x_2^2x_3^2x_4^2 \,|\, x_2x_3x_4 \,| \,\}.$$

Now we have $\quad \Sigma \,|\, x_2x_3x_4 \,| = 0, \quad \Sigma (x_2 + x_3 + x_4) \,|\, x_2x_3x_4 \,| = 0,$

$\Sigma (x_2 + x_3 + x_4)^2 \,|\, x_2x_3x_4 \,| = \Sigma (x_2^2 + x_3^2 + x_4^2) \,|\, x_2x_3x_4 \,| + 2\Sigma (x_2x_3 + x_3x_4 + x_4x_2 \,|\, x_2x_3x_4 \,|$

it is easy to see that $\Sigma (x_2^2 + x_3^2 + x_4^2) \,|\, x_2x_3x_4 \,| = 0$, and we know that

$$\Sigma (x_2x_3 + x_3x_4 + x_4x_2) \,|\, x_2x_3x_4 \,| = 0,$$
$$\therefore \quad \Sigma (x_2 + x_3 + x_4)^2 \,|\, x_2x_3x_4 \,| = 0,$$

also

$$\Sigma (x_2x_3 + x_3x_4 + x_4x_2)^2 \,|\, x_2x_3x_4 \,| = 0,$$
$$\Sigma x_2x_3x_4 \,|\, x_2x_3x_4 \,| = -\,\theta,$$
$$\Sigma x_2x_3x_4 (x_2 + x_3 + x_4) \,|\, x_2x_3x_4 \,| = -\,\alpha\theta,$$
$$\Sigma x_2x_3x_4 (x_2x_3 + x_3x_4 + x_4x_2) \,|\, x_2x_3x_4 \,| = -\,\beta\theta,$$
$$\Sigma x_2^2x_3^2x_4^2 \,|\, x_2x_3x_4 \,| = \gamma\theta.$$

We have now after easy reductions

$$\pi\theta \{ a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta^* \}$$
$$- \pi\theta \{ a_l^4 - \alpha a_l^3 + \beta a_l^2 - \gamma a_l + \delta^* \},$$

the δ^* being introduced simply for symmetry. In the following the terms marked by the asterisk are all introduced simply for symmetry, and obviously

cancel each other. Making reductions for the remaining terms similar to those just made for the term containing π we find

$$\begin{aligned} |a_k a_l| \Sigma |x_2 x_3 x_4| & [a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4] \\ = |x_1 x_2 x_3 x_4| \{ & \pi(a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta^*) \\ & - \pi(a_l^4 - \alpha a_l^3 + \beta a_l^2 - \gamma a_l + \delta^*) \\ & - \nu a_l(a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k^* + \delta) \\ & + \nu a_k(a_l^4 - \alpha a_l^3 + \beta a_l^2 - \gamma a_l^* + \delta) \\ & + \mu a_l^2(a_k^4 - \alpha a_k^3 + \beta a_k^{2*} - \gamma a_k + \delta) \\ & - \mu a_k^2(a_l^4 - \alpha a_l^3 + \beta a_l^{2*} - \gamma a_l + \delta) \\ & - \lambda a_l^3(a_k^4 - \alpha a_k^{3*} + \beta a_k^2 - \gamma a_k + \delta) \\ & + \lambda a_k^3(a_l^4 - \alpha a_l^{3*} + \beta a_l^2 - \gamma a_l + \delta) \\ & + a_l^4(a_k^{4*} - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta) \\ & - a_k^4(a_l^{4*} - \alpha a_l^3 + \beta a_l^2 - \gamma a_l + \delta) \} \end{aligned}$$

or
$$\begin{aligned} = |x_1 x_2 x_3 x_4| \{ & (\pi - \nu a_l + \mu a_l^2 - \lambda a_l^3 + a_l^4)(a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta) \\ & - (\pi - \nu a_k + \mu a_k^2 - \lambda a_k^3 + a_k^4)(a_l^4 - \alpha a_l^3 + \beta a_l^2 - \gamma a_l + \delta) \}. \end{aligned}$$

From equation 9 however we have

$$a_k^4 - \alpha a_k^3 + \beta a_k^2 - \gamma a_k + \delta = P_k^2, = |a_k x_1||a_k x_2||a_k x_3||a_k x_4|$$
$$a_l^4 - \alpha a_l^3 + \beta a_l^2 - \gamma a_l + \delta = P_l^2, = |a_l x_1||a_l x_2||a_l x_3||a_l x_4|$$

similarly we have

$$\pi - \nu a_l + \mu a_l^2 - \lambda a_l^3 + a_l^4 = |a_q a_l||a_r a_l||a_s a_l||a_t a_l|$$
$$\pi - \nu a_k + \mu a_k^2 - \lambda a_k^3 + a_k^4 = |a_q a_k||a_r a_k||a_s a_k||a_t a_k|$$

and therefore finally

48.
$$\begin{aligned} & |a_k a_l| \{ a_n a_p | P_{klm}^2 + |a_p a_m| P_{kln}^2 + |a_m a_n| P_{klp}^2 \} \\ & = |a_m a_n a_p| \{ |a_k a_l| P_{kl}^2 + |a_q a_l||a_r a_l||a_s a_l||a_t a_l| P_k^2 - |a_q a_k||a_r a_k||a_s a_k||a_t a_k| P_l^2 \}. \end{aligned}$$

Effect upon this the substitution

$$\begin{vmatrix} m & n & p & q & r & s & t \\ n & p & q & r & s & t & m \end{vmatrix}$$

and we arrive at a new relation

49.
$$\begin{aligned} & |a_k a_l| \{ |a_p a_q| P_{kln}^2 + |a_q a_n| P_{klp}^2 + |a_n a_p| P_{klq}^2 \} \\ & = |a_n a_p a_q| \{ |a_k a_l| P_{kl}^2 + |a_r a_l||a_s a_l||a_t a_l||a_m a_l| P_k^2 - |a_r a_k||a_s a_k||a_t a_k||a_m a_k| P_l^2 \}. \end{aligned}$$

Eliminating P_{kl}^2 between 48 and 49 and we find a linear relation connecting

$$P_{klm}^2, \; P_{kln}^2, \; P_{klp}^2, \; P_{klq}^2, \; P_{kl}^2, \; P_k^2,$$

viz., after dropping out a common factor $|a_n a_p|$,

50.
$$\begin{aligned} & |a_k a_l| \{ |a_n a_p a_q| P_{klm}^2 - |a_p a_q a_m| P_{kln}^2 + |a_q a_m a_n| P_{klp}^2 - |a_m a_n a_p| P_{klq}^2 \} \\ & = |a_m a_n a_p a_q| \{ |a_r a_k||a_s a_k||a_t a_k| P_l^2 - |a_r a_l||a_s a_l||a_t a_l| P_k^2 \}. \end{aligned}$$

If instead of eliminating P^2_{kl} between 48 and 49 we had eliminated P^2_l, we would arrive at a linear relation connecting

$$P^2_{klm},\ P^2_{kln},\ P^2_{klp},\ P^2_{klq},\ P^2_{kl},\ P^2_k,$$

say

51. $$AP^2_{klm} + BP^2_{kln} + CP^2_{klp} + DP^2_{klq} + EP^2_{kl} + FP^2_k = 0.$$

Now effect upon 50 the substitution

$$\left| \begin{matrix} m & n & p & q & r & s & t \\ n & p & q & r & s & t & m \end{matrix} \right| = \text{say } \Omega,$$

then we will obviously obtain a relation of the form,

52. $$A'P^2_{kln} + B'P^2_{klp} + C'P^2_{klq} + D'P_{klr} + E'P^2_{kl} + F'P^2_k = 0.$$

Eliminating P^2_{kl} between 51 and 52 and we have a linear relation connecting five of the P-functions possessing triple indices, and one possessing a single index, viz., a relation of the form

53. $$A''P^2_{klm} + B''P^2_{kln} + C''P^2_{klp} + D''P^2_{klq} + E''P^2_{klr} + F''P^2_k = 0.$$

Or, if P^2_k had been eliminated, we would have a relation connecting five P-functions with triple indices, and one with a double index, viz.

54. $$A'''P^2_{klm} + B'''P^2_{kln} + C'''P^2_{klp} + D'''P^2_{klq} + E'''P^2_{klr} + F'''P^2_{kl} = 0.$$

Effecting the substitution Ω upon 54 and there results an equation of the form,

55. $$A^{\text{iv}}P^2_{kln} + B^{\text{iv}}P^2_{klp} + C^{\text{iv}}P^2_{klq} + D^{\text{iv}}P^2_{klr} + E^{\text{iv}}P^2_{kls} + F^{\text{iv}}P^2_{kl} = 0.$$

Now finally eliminating P^2_{kl} between 54 and 55 and we arrive at a linear relation connecting the squares of six P-functions with triple indices, viz.

56. $$A^{\text{v}}P^2_{klm} + B^{\text{v}}P^2_{kln} + C^{\text{v}}P^2_{klp} + D^{\text{v}}P^2_{klq} + E^{\text{v}}P^2_{klr} + F^{\text{v}}P^2_{kls} = 0.$$

Of course, the substitution Ω performed upon 56 would give

57. $$A^{\text{vi}}P^2_{kln} + B^{\text{vi}}P^2_{klp} + C^{\text{vi}}P^2_{klq} + D^{\text{vi}}P^2_{klr} + E^{\text{vi}}P^2_{kls} + F^{\text{vi}}P^2_{klt} = 0.$$

We arrive thus at the conclusion that the ten functions

$$P_k,\ P_l,\ P_{kl},\ P_{klm},\ P_{kln},\ P_{klp},\ P_{klq},\ P_{klr},\ P_{kls},\ P_{klt}$$

form a group such that selecting any five of them the square of any one of the remaining five is linearly expressible in terms of the squares of the chosen five. There are in all 36 such groups, and they are given in the following table:

k	l	kl	klm	kln	klp	klq	klr	kls	klt
k	m	km	kml	kmn	kmp	kmq	kmr	kms	kmt
k	n	kn	knl	knm	knp	knq	knr	kns	knt
k	p	kp	kpl	kpm	kpn	kpq	kpr	kps	kpt
k	q	kq	kql	kqm	kqn	kqp	kqr	kqs	kqt
k	r	kr	krl	krm	krn	krp	krq	krs	krt
k	s	ks	ksl	ksm	ksn	ksp	ksq	ksr	kst

k	t	kt	ktl	ktm	ktn	ktp	ktq	ktr	kts
l	m	lm	lmk	lmn	lmp	lmq	lmr	lms	lmt
l	n	ln	lnk	lnm	lnp	lnq	lnr	lns	lnt
l	p	lp	lpk	lpm	lpn	lpq	lpr	lps	lpt
l	q	lq	lqk	lqm	lqn	lqp	lqr	lqs	lqt
l	r	lr	lrk	lrm	lrn	lrp	lrq	lrs	lrt
l	s	ls	lsk	lsm	lsn	lsp	lsq	lsr	lst
l	t	lt	ltk	ltm	ltn	ltp	ltq	ltr	lts
m	n	mn	mnk	mnl	mnp	mnq	mnr	mns	mnt
m	p	mp	mpk	mpl	mpn	mpq	mpr	mps	mpt
m	q	mq	mqk	mql	mqn	mqp	mqr	mqs	mqt
m	r	mr	mrk	mrl	mrn	mrp	mrq	mrs	mrt
m	s	ms	msk	msl	msn	msp	msq	msr	mst
m	t	mt	mtk	mtl	mtn	mtp	mtq	mtr	mts
n	p	np	npk	npl	npm	npq	npr	nps	npt
n	q	nq	nqk	nql	nqm	nqp	nqr	nqs	nqt
n	r	nr	nrk	nrl	nrm	nrp	nrq	nrs	nrt
n	s	ns	nsk	nsl	nsm	nsp	nsq	nsr	nst
n	t	nt	ntk	ntl	ntm	ntp	ntq	ntr	nts
p	q	pq	pqk	pql	pqm	pqn	pqr	pqs	pqt
p	r	pr	prk	prl	prm	prn	prq	prs	prt
p	s	ps	psk	psl	psm	psn	psq	psr	pst
p	t	pt	ptk	ptl	ptm	ptn	ptq	ptr	pts
q	r	qr	qrk	qrl	qrm	qrn	qrp	qrs	qrt
q	s	qs	qsk	qsl	qsm	qsn	qsp	qsr	qst
q	t	qt	qtk	qtl	qtm	qtn	qtp	qtr	qts
r	s	rs	rsk	rsl	rsm	rsn	rsp	rsq	rst
r	t	rt	rtk	rtl	rtm	rtn	rtp	rtq	rts
s	t	st	stk	stl	stm	stn	stp	stq	str

Consider now the functions with four indices and find values of A, B, C, D, such that the radicals $\sqrt{R(x_1)R(x_2)}$, etc. shall vanish in the sum

$$AP^2_{klmn} - BP^2_{klmp} + CP^2_{klmq} - DP^2_{klmr}.$$

The coefficient of $\sqrt{R(x_1)R(x_2)}$ in this sum is, leaving out the common factor $|x_2x_3x_4||x_3x_4x_1|$,

$$A[a_k, a_l, a_m, a_n][x_3][a_k, a_l, a_m, a_n][x_4] - B[a_k, a_l, a_m, a_p][x_3][a_k, a_l, a_m, a_p][x_4]$$
$$+ C[a_k, a_l, a_m, a_q][x_3][a_k, a_l, a_m, a_q][x_4] - D[a_k, a_l, a_m, a_r][x_3][a_k, a_l, a_m, a_r][x_4] = 0.$$

Taking out the common factor

$$[a_k, a_l, a_m][x_3][a_k, a_l, a_m][x_4],$$

and this becomes

$$A\,|a_n x_3||a_n x_4| - B\,|a_p x_3||a_p x_4| + C\,|a_q x_3||a_q x_4| - D\,|a_r x_3||a_r x_4| = 0,$$

giving

$$A = |a_p a_q a_r|, \quad B = |a_q a_r a_n|,$$
$$C = |a_r a_n a_p|, \quad D = |a_n a_p a_q|.$$

Introducing here the values

$$R(x_1) = [a_k, a_l, a_m, a_n, a_p, a_q, a_r, a_s, a_t][x_1], \text{ etc.}$$

we ought to be able to show that the squares of any six of the functions whose indices are *kl lm mk klm klmn klmp klmq klmr klms klmt* are connected by a linear relation. We would then have a table of 84 groups similar to the above, and such that the squares of any six functions in a given group are connected by a linear relation. In the case of quadruple indices there would also be a second table containing 126 groups, of which

klm lmn mnk nkl klmn pqrs qrst rstp stpq tpqr

is the first, and the squares of any six of these functions should also be connected by a linear relation.

We would thus have in all 256 groups giving linear relations between the squares of six P-functions. There would be 840 relations of this kind, but not all, of course, asyzygetic.

These 256 groups of ten functions each might be called the 256 decads, and they would obviously correspond to the 16 Kummer hexads in the case of the double theta-functions, viz. between the squares of any four theta-functions of a Kummer hexad there exists a linear relation, and between the squares of any six of the P-functions belonging to a given decad there exists a linear relation, the same kind of relation will obviously exist between the squares of the six corresponding quadruple theta-functions. We have thus hexads of double theta-functions of which the squares of any four are connected by a linear relation; octads of triple theta-functions of which the squares of any five are connected by a linear relation; decads of quadruple theta-functions of which the squares of any six are connected by a linear relation, and in general $2(p+1)$-ads of p-tuple theta-functions of which the squares of any $p+2$ are connected by a linear relation. It seems highly probable that this generalization is true, but I have not as yet been able to prove it.

In order to show the linear relations between the squares of the P-functions with quadruple indices, we will begin with the 84 groups, of which

$$kl\ lm\ mk\ klm\ klmn\ klmp\ klmq\ klmr\ klms\ klmt$$

is the first. Take the functions

$$P_{klm},\ P_{klmn},\ P_{klmp},\ P_{klmq},$$

and find the values of A, B, C, D, so that the radicals $\sqrt{R(x_1)R(x_2)}$, etc., shall vanish in the sum $AP^2_{klm} + BP^2_{klmn} + CP^2_{klmp} + DP^2_{klmq}$.

Dropping out a common factor

$$[a_k][x_3,\ x_4][a_l][x_3,\ x_4][a_m][x_3,\ x_4]$$

the necessary condition is easily seen to be

$$A + B[a_n][x_3,\ x_4] + C[a_p][x_3,\ x_4] + D[a_q][x_3,\ x_4] = 0;$$

this is equivalent to

$$B + C + D = 0,$$
$$a_n B + a_p C + a_q D = 0,$$
$$A + a_n^2 B + a_p^2 C + a_q^2 D = 0,$$

and these are easily seen to be satisfied by the values

$$A = -|a_n a_p a_q|,\quad B = |a_p a_q|,\quad C = |a_q a_n|,\quad D = |a_n a_p|.$$

Introducing these values in the above sum we have

$$-|a_n a_p a_q| P^2_{klm} + |a_p a_q| P^2_{klmn} + |a_q a_n| P^2_{klmp} + |a_n a_p| P^2_{klmq}$$

$$= \frac{1}{|x_1x_2x_3x_4|^2}$$

$$\{-|a_na_pa_q|[|x_2x_3x_4|^2[a_n,\ a_p,\ a_q,\ a_r,\ a_s,\ a_t][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4] + ..]$$
$$+|a_pa_q|[|x_2x_3x_4|^2[a_p, a_q, a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4][a_n][x_2, x_3, x_4] + ..]$$
$$+|a_qa_n|[|x_2x_3x_4|^2[a_q, a_r, a_s, a_t, a_n][x_1] \text{ “ “ “ “ “ “ “ “ }] + ..]$$
$$+|a_na_p|[|x_2x_3x_4|^2[a_r, a_s, a_t, a_n, a_p][x_1] \text{ “ “ “ “ “ “ “ “ }] + ..]\}$$

The right-hand side of this equation may be written in the form

$$\frac{1}{|x_1x_2x_3x_4|^2}[|a_pa_q||x_2x_3x_4|[a_r,\ a_s,\ a_t,\ a_p,\ a_q][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4]$$
$$\times\{|x_2x_3x_4|[a_n][x_2,\ x_3,\ x_4] - |x_3x_4x_1|[a_n][x_3,\ x_4,\ x_1] + |x_4x_1x_2|[a_n][x_2,\ x_3,\ x_4]$$
$$-|x_1x_2x_3|[a_n][x_1,\ x_2,\ x_3]\} + \ldots$$
$$+ \text{ (two similar terms containing } [a_q,\ a_n][x_1],\ [a_n,\ a_p][x_1] \text{ respectively)}$$
$$-|a_n a_p a_q|[a_r,\ a_s,\ a_t,\ a_n,\ a_p,\ a_q][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4]$$
$$\times\{|x_2x_3x_4| - |x_3x_4x_1| + |x_4x_1x_2| - |x_1x_2x_3|\} + \ldots]$$

The omitted terms are easily supplied by symmetry. Now

$$\Sigma|x_2x_3x_4| = 0,\quad \Sigma|x_2x_3x_4|[a_n][x_2,\ x_3,\ x_4] = |x_1x_2x_3x_4|$$

The fourth line of the last equation vanishes and the first terms of the first three

$$|x_2x_3x_4|[a_r,\ a_s,\ a_t][x_1][a_k][x_2,\ x_3,\ x_4][a_l][x_2,\ x_3,\ x_4][a_m][x_2,\ x_3,\ x_4]$$

for a common factor; similar factors exist of course for the remaining nine terms (taken three at a time) of the first three lines. This factor is to be multiplied by

$$|a_p a_q||a_p x_1||a_q x_1| + |a_q a_n||a_q x_1||a_n x_1| + |a_n a_p||a_n x_1||a_p x_1|, = |a_n a_p a_q|$$

We have then for the right-hand side of 58 the value

59. $\dfrac{1}{|x_1x_2x_3x_4|}|a_n a_p a_q|\Sigma|x_2x_3x_4|[a_r, a_s, a_t][x_1][a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4]$

It is quite easy to show that the quantity under the sign of summation contains θ, *i. e.* $|x_1x_2x_3x_4|$ as a factor. In fact replacing a_k, a_l, a_m by a, b, c respectively and writing

$$\alpha_1 = x_2 + x_3 + x_4,$$
$$\beta_1 = x_2x_3 + x_3x_4 + x_4x_2,$$
$$\gamma_1 = x_2x_3x_4,$$

with similar expressions for α_2, β_2, γ_2, etc., obtained by advancing the suffixes of the x's, we have for the expansion of $[a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4]$ the following

1	$+\,a^3b^3c^3$		$\beta_1\gamma_1$	$-\,\Sigma ab^3$
α_1	$-\,\Sigma a^2b^3c^3$		$\alpha_1^2\gamma_1$	$-\,\Sigma a^2b^2$
α_1^2	$+\,\Sigma a^2b^2c^3$		$\alpha_1\beta_1^2$	$-\,\Sigma abc^2$
β_1	$+\,\Sigma ab^3c^3$		β_1^3	$+\,abc$
α_1^3	$-\,a^2b^2c^2$		γ_1^2	$+\,\Sigma a^3$
γ_1	$-\,\Sigma b^3c^3$	+	$\alpha_1\beta_1\gamma_1$	$+\,\Sigma ab^2$
$\alpha_1\beta_1$	$-\,\Sigma ab^2c^3$		$\beta_1^2\gamma_1$	$-\,\Sigma ab$
β_1^2	$+\,\Sigma abc^3$		$\alpha_1\gamma_1^2$	$-\,\Sigma a^2$
$\alpha_1\gamma_1$	$+\,\Sigma a^2b^3$		$\beta_1\gamma_1^2$	$+\,\Sigma a$
$\alpha_1^2\beta_1$	$+\,\Sigma ab^2c^2$		γ_1^3	$-\,1$

The second columns here are the coefficients of the quantities in the first columns. Form now the sum

$$\Sigma|x_2x_3x_4|[a_k][x_2, x_3, x_4][a_l][x_2, x_3, x_4][a_m][x_2, x_3, x_4].$$

Neglecting the coefficients depending upon a, b, c, we have the identities

$$\Sigma\,|x_2x_3x_4| = 0 \qquad \Sigma\alpha_1\gamma_1\,|x_2x_3x_4| = \alpha\theta$$
$$\Sigma\alpha_1\,|x_2x_3x_4| = 0 \qquad \Sigma\alpha_1^2\beta_1\,|x_2x_3x_4| = -\alpha\theta$$
$$\Sigma\beta_1\,|x_2x_3x_4| = 0 \qquad \Sigma\beta_1\gamma_1\,|x_2x_3x_4| = -\beta\theta$$
$$\Sigma\alpha_1^2\,|x_2x_3x_4| = 0 \qquad \Sigma\alpha_1^2\gamma_1\,|x_2x_3x_4| = -\beta\theta$$
$$\Sigma\alpha_1^3\,|x_2x_3x_4| = -\theta \qquad \Sigma\alpha_1\beta_1^2\,|x_2x_3x_4| = \beta\theta$$
$$\Sigma\gamma_1\,|x_2x_3x_4| = -\theta \qquad \text{and so on to}$$
$$\Sigma\alpha_1\beta_1\,|x_2x_3x_4| = -\theta \qquad \Sigma\gamma_1^3\,|x_2x_3x_4| = -(\gamma^2 - 2\beta\delta)\,\theta$$
$$\Sigma\beta_1^2\,|x_2x_3x_4| = -\alpha\theta$$

It is obvious then that θ is a factor of the quantity under the summation sign in 59. Beyond this point I have not, as yet, been able to go. On reducing 59, we ought, if the general theorem stated above be true, to find that it is equal to the sum of the squares of two of the functions P_{kl}, P_{lm}, P_{mk}, each multiplied by a constant coefficient depending upon the quantities a. But from the symmetry of the thing we can see *à priori* that there is no reason why any particular pair of these functions should come in more than any other pair. I have made two or three reductions of 59, but have not as yet been able to arrive at any interpretable result.

On certain Groups of Relations satisfied by the Quadruple Theta-Functions.

By Thomas Craig, *Johns Hopkins University.*

If m_1, m_2, m_3, m_4 denote even integers, positive or negative, and if we write

$$\begin{pmatrix} m_1+\alpha_1, & m_2+\alpha_2, & m_3+\alpha_3, & m_4+\alpha_4 \\ u_1+\beta_1, & u_2+\beta_2, & u_3+\beta_3, & u_4+\beta_4 \end{pmatrix}$$
$$= \tfrac{1}{4}(a_{11}, a_{12}, \ldots a_{44})(m_1+\alpha_1, \ldots m_4+\alpha_4)^2$$
$$+ \tfrac{1}{2}\pi i\left[(m_1+\alpha_1)(u_1+\beta_1)+\ldots+(m_4+\alpha_4)(u_4+\beta_4)\right]$$

we have for the definition of the quadruple theta-functions the equation

$$\vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(u_1u_2u_3u_4) =$$
$$\sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4}\exp.\begin{pmatrix} m_1+\alpha_1, & m_2+\alpha_2, & m_3+\alpha_3, & m_4+\alpha_4 \\ u_1+\beta_1, & u_2+\beta_2, & u_3+\beta_3, & u_4+\beta_4 \end{pmatrix}$$

or briefly
$$\vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(u) = \Sigma\exp.\begin{pmatrix}m+\alpha\\ u+\beta\end{pmatrix},$$
where the summations extend over all positive and negative even integer values of m_1, m_2, m_3 and m_4. Suppose now we write

$$a_{11} = \log q_1,\quad a_{22} = \log q_2,\quad a_{33} = \log q_3,\quad a_{44} = \log q_4,$$
$$a_{12} = \log q_{12},\quad a_{13} = \log q_{13},\quad a_{14} = \log q_{14},\quad a_{23} = \log q_{23},\ \text{etc.}$$

and also
$$u_1 = \frac{v_1}{K_1},\quad u_2 = \frac{v_2}{K_2},\quad u_3 = \frac{v_3}{K_3},\quad u_4 = \frac{v_4}{K_4},$$

and replace the m_1, m_2, m_3, m_4 by $2m_1$, etc.; then the summations will extend over all positive and negative values of the m's, and it is easy to see that we have

$$\vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(u_1u_2u_3u_4) =$$
$$\sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4}(-)^{\Sigma m\beta}\exp.\Big\{\tfrac{1}{4}\left[(2m_1+\alpha_1)^2\log q_1+\ldots+(2m_4+\alpha_4)^2\log q_4\right.$$
$$+2(2m_1+\alpha_1)(2m_2+\alpha_2)\log q_{12}$$
$$\left.+\ldots+2(2m_3+\alpha_3)(2m_4+\alpha_4)\log q_{34}\right]$$
$$+\frac{i\pi}{2}\left((2m_1+\alpha_1)\frac{v_1}{K_1}+\ldots+(2m_4+\alpha_4)\frac{v_4}{K_4}\right)\Big\}.$$

The true periods are obviously

v_1	$4K_1$	0	0	0
v_2	0	$4K_2$	0	0
v_3	0	0	$4K_3$	0
v_4	0	0	0	$4K_4$

The quasi-periods are easily written down, but it is not worth while to give them here.

Write the exponent in the following manner:

$$\tfrac{1}{4}(2m_1+\alpha_1)^2\log q_1+\frac{i\pi v_1}{2K}(2m_1+\alpha_1)$$
$$+\{\tfrac{1}{4}(2m_2+\alpha_2)^2\log q_2+\tfrac{1}{4}(2m_3+\alpha_3)^2\log q_3+\tfrac{1}{4}(2m_4+\alpha_4)^2\log q_4$$
$$+2(2m_1+\alpha_1)(2m_2+\alpha_2)\log q_{12}+2(2m_1+\alpha_1)(2m_3+\alpha_3)\log q_{13}$$
$$+2(2m_1+\alpha_1)(2m_4+\alpha_4)\log q_{14}$$
$$+2(2m_2+\alpha_2)(2m_3+\alpha_3)\log q_{23}+2(2m_2+\alpha_2)(2m_4+\alpha_4)\log q_{24}$$
$$+2(2m_3+\alpha_3)(2m_4+\alpha_4)\log q_{34}$$
$$+\frac{i\pi v_2}{2K_2}(2m_2+\alpha_2)+\frac{i\pi v_3}{2K_3}(2m_3+\alpha_3)+\frac{i\pi v_4}{2K_4}(2m_4+\alpha_4)\}.$$

Write now

$$\log q_{12}=\frac{i\pi}{2K_2}\log p_{12},\ \log q_{13}=\frac{i\pi}{2K_3}\log p_{13},\ \log q_{14}=\frac{i\pi}{2K_4}\log p_{14}$$

and

$$v_2=w_2-\frac{2m_1+\alpha_1}{2}\log p_{12},\ v_3=w_3-\frac{2m_1+\alpha_1}{2}\log p_{13},\ v_4=w_4-\frac{2m_1+\alpha_1}{2}\log p_{14}.$$

Taking the terms which contain $\log q_{12}$ and v_2 and combine them after making these substitutions and we have

$$\tfrac{1}{4}\,.\,2(2m_1+\alpha_1)(2m_2+\alpha_2)\log q_{12}+\frac{i\pi v_2}{2K_2}(2m_2+\alpha_2)$$
$$=\tfrac{1}{4}\,.\,2(2m_1+\alpha_1)(2m_2+\alpha_2)\frac{i\pi}{2K_2}\log p_{12}+(2m_2+\alpha_2)\left(\frac{i\pi w_2}{2K_2}-\frac{i\pi}{2K_2}\cdot\frac{2m_1+\alpha_1}{2}\log p_{12}\right)$$
$$=\frac{i\pi w_2}{2K_2}(2m_2+\alpha_2).$$

Similar results are obtained for the terms containing $\log q_{13}$ and v_3, $\log q_{14}$ and v_4. Combining all these and the new exponent is:

$$\tfrac{1}{4}(2m_1+\alpha_1)^2\log q_1+\frac{i\pi v_1}{2K_1}(2m_1+\alpha_1)$$

$$+\Big\{\tfrac{1}{4}(2m_2+\alpha_2)^2\log q_2+\tfrac{1}{4}(2m_3+\alpha_3)^2\log q_3+\tfrac{1}{4}(2m_4+\alpha_4)^2\log q_4$$
$$+2(2m_2+\alpha_2)(2m_3+\alpha_3)\log q_{23}+2(2m_2+\alpha_2)(2m_4+\alpha_4)\log q_{24}$$
$$+2(2m_3+\alpha_3)(2m_4+\alpha_4)\log q_{34}$$
$$+\frac{i\pi w_2}{2K_2}(2m_2+\alpha_2)+\frac{i\pi w_3}{2K_3}(2m_3+\alpha_3)+\frac{i\pi w_4}{2K_4}(2m_4+\alpha_4)\Big\}.$$

The terms in $\{\ \}$ obviously form the exponent of a triple theta-function, viz.

$$\vartheta\begin{pmatrix}\alpha_2\,\alpha_3\,\alpha_4\\ \beta_2\,\beta_3\,\beta_4\end{pmatrix}(w_2w_3w_4),\ \text{or simply}\ \vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(w)$$

understanding that here the suffixes are 2, 3, 4. We have then finally

$$1.\ \vartheta\begin{pmatrix}\alpha_1\alpha_2\alpha_3\alpha_4\\ \beta_1\beta_2\beta_3\beta_4\end{pmatrix}(v_1v_2v_3v_4)=\sum_{m_1}(-)^{m_1\beta_1}e^{\frac{1}{4}(2m_1+\alpha_1)^2\log q_1+\frac{i\pi v_1}{2K_1}(2m_1+\alpha_1)}.\vartheta\begin{pmatrix}\alpha_2\alpha_3\alpha_4\\ \beta_2\beta_3\beta_4\end{pmatrix}(w_2w_3w_4)$$

For brevity write this in the form

$$2.\qquad \Theta_4(v_1v_2v_3v_4)=\sum_{m_1}(-)^{m_1\beta_1},\ q_1^{\left(m_1+\frac{\alpha_1}{2}\right)^2}e^{\frac{i\pi v_1}{K_1}\left(m_1+\frac{\alpha_1}{2}\right)}.\Theta_3(w_2w_3w_4),$$

or substituting for w_2, w_3, w_4 their values

$$3.\quad \Theta_4(v_1v_2v_3v_4)=\sum_{m_1}(-)^{m_1\beta_1}q_1^{\left(m_1+\frac{\alpha_1}{2}\right)^2}e^{\frac{i\pi v_1}{K_1}\left(m_1+\frac{\alpha_1}{2}\right)}$$
$$\Theta_3\left(v_2+\frac{2m_1+\alpha_1}{2}\log p_{12},\ v_3+\frac{2m_1+\alpha_1}{2}\log p_{13},\ v_4+\frac{2m_1+\alpha_1}{2}\log p_{14}\right),$$

the summation extending from $m_1=-\infty$ to $m_1=+\infty$. Expand this by giving to m_1 all of its values and grouping together the terms corresponding to equal positive and negative values of m_1,

$$4.\qquad \Theta_4(v_1v_2v_3v_4)=q_1^{\frac{\alpha_1^2}{4}}e^{\frac{i\pi v_1}{K_1}\cdot\frac{\alpha_1}{2}}.\Theta_3\left(v_2+\frac{\alpha_1}{2}\log p_{12},\ v_3+\frac{\alpha_1}{2}\log p_{13},\ v_4+\frac{\alpha_1}{2}\log p_{14}\right)$$
$$+(-)^{\beta_1}q_1^{\frac{(2+\alpha_1)^2}{4}}\Big\{\cos(2+\alpha_1)\tau\Big[\Theta_3\Big(v_2+\frac{2+\alpha_1}{2}\log p_{12}\ldots\Big)$$
$$+q^{-4\alpha_1}(\cos 2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\Big(v_2-\frac{2-\alpha_1}{2}\log p_{12}\ldots\Big)\Big]$$
$$+i\sin(2+\alpha_1)\tau\Big[\Theta_3\Big(v_2+\frac{2+\alpha_1}{2}\log p_{12}\ldots\Big)$$
$$-q^{-4\alpha_1}\cos(2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\Big(v_2-\frac{2-\alpha_1}{2}\log p_{12}\ldots\Big)\Big]\Big\}$$
$$+(-)^{2\beta_1}q^{\frac{(4+\alpha_1)^2}{4}}\Big\{\cos(4+\alpha_1)\tau\Big[\Theta_3\Big(v_2+\frac{4+\alpha_1}{2}\log p_{12}\ldots\Big)$$
$$+q^{-16\alpha_1}(\cos 2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\Big(v_2-\frac{4-\alpha_1}{2}\log p_{12}\ldots\Big)\Big]$$
$$+i\sin(4+\alpha_1)\tau\Big[\Theta_3\Big(v_2+\frac{4+\alpha_1}{2}\log p_{12}\ldots\Big)$$
$$-q^{-16\alpha_1}(\cos 2\alpha_1\tau+i\sin 2\alpha_1\tau)\Theta_3\Big(v_3-\frac{4-\alpha_1}{2}\log p_{12}\ldots\Big)\Big]\Big\}$$
$$+\text{etc.}$$

where for brevity I have written $\tau_1 = \frac{\pi v_1}{2K_1}$.

The single theta-function $\vartheta\binom{\alpha_1}{\beta_1}(v_1)$ on being expanded takes the form

$$5.\quad \vartheta\binom{\alpha_1}{\beta_1}(v_1) = q_1^{\frac{\alpha_1^2}{4}} e^{\alpha_1\tau_1} + (-)^{\beta_1} q_1^{\frac{(2+\alpha_1)^2}{4}} \{\cos(2+\alpha_1)\tau_1[1+q_1^{-4\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]$$
$$+ i\sin(2+\alpha_1)\tau_1[1-q_1^{-4\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]\}$$
$$+ (-)^{2\beta_1} q_1^{\frac{(4+\alpha_1)^2}{4}} \{\cos(4+\alpha_1)\tau_1[1+q^{-16\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]$$
$$+ i\sin(4+\alpha_1)\tau_1[1-q_1^{-16\alpha_1}(\cos 2\alpha_1\tau_1 + i\sin 2\alpha_1\tau_1)]\}$$
$$+ \text{etc.}$$

Giving $\binom{\alpha_1}{\beta_1}$ the values $\binom{0}{0}$, $\binom{1}{0}$, $\binom{0}{1}$, $\binom{1}{1}$ this general form gives the four known equations

$$\vartheta\binom{0}{0}(v_1) = 1 + 2q_1 \cos 2\frac{\pi v_1}{2K_1} + 2q_1^4 \cos 4\frac{\pi v_1}{2K_1} + 2q_1^9 \cos 6\frac{\pi v_1}{2K_1} + \ldots$$
$$\vartheta\binom{1}{0}(v_1) = 1 - 2q_1 \cos 2\frac{\pi v_1}{2K_1} + 2q_1^4 \cos 4\frac{\pi v_1}{2K_1} - 2q_1^9 \cos 6\frac{\pi v_1}{2K_1} + \ldots$$
$$\vartheta\binom{0}{1}(v_1) = 2q_1^{\frac{1}{4}} \cos\frac{\pi v_1}{2K_1} + 2q_1^{\frac{9}{4}} \cos 3\frac{\pi v_1}{2K_1} + 2q_1^{\frac{25}{4}} \cos 5\frac{\pi v_1}{2K_1} + \ldots$$
$$-i\vartheta\binom{1}{1}(v_1) = 2q_1^{\frac{1}{4}} \sin\frac{\pi v_1}{2K_1} - 2q_1^{\frac{9}{4}} \cos 3\frac{\pi v_1}{2K_1} + 2q_1^{\frac{25}{4}} \sin 5\frac{\pi v_1}{2K_1} + \ldots$$

For convenience write $\vartheta\binom{\alpha_1}{\beta_1}(v_1)$ as simply $\Theta_1(v_1)$ and bear in mind that

$$\Theta_4(v_1 v_2 v_3 v_4) \equiv \vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_1 v_2 v_3 v_4)$$

and
$$\Theta_3(v_2 v_3 v_4) \equiv \vartheta\begin{pmatrix}\alpha_2\,\alpha_3\,\alpha_4\\ \beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_2 v_3 v_4).$$

Now by aid of 5, equation 4 may be thrown into the form

$$6.\qquad \Theta_4(v_1, v_2, v_3, v_4) = \Theta_3(v_2, v_3, v_4).\Theta_1(v_1)$$
$$-\frac{K_1}{\pi}\frac{d\theta_1(v_1)}{dv_1}\left\{\frac{2K_2}{\pi}\log q_{12}\frac{d}{dv_2} + \frac{2K_3}{\pi}\log q_{13}\frac{d}{dv_3} + \frac{2K_4}{\pi}\log q_{14}\frac{d}{dv_4}\right\}\Theta_3(v_2, v_3, v_4)$$
$$+\frac{1}{2}\left(\frac{K_1}{\pi}\right)^2\frac{d^2\theta_1(v_1)}{dv_1^2}\left\{\frac{2K_2}{\pi}\log q_{12}\frac{d}{dv_2} + \frac{2K_3}{\pi}\log q_{13}\frac{d}{dv_3} + \frac{2K_4}{\pi}\log q_{14}\frac{d}{dv_4}\right\}\Theta_3(v_2, v_3, v_4)$$
$$-\frac{1}{3}\left(\frac{K_1}{\pi}\right)^3\frac{d^3\theta(v_1)}{dv_1^3}\left\{\frac{2K_2}{\pi}\log q_{12}\frac{d}{dv_2} + \frac{2K_3}{\pi}\log q_{13}\frac{d}{dv_3} + \frac{2K_4}{\pi}\log q_{14}\frac{d}{dv_4}\right\}\Theta_3(v_2, v_3, v_4)$$
$$+ \text{etc.},$$

or symbolically this may be written,

$$7.\qquad \Theta_4 = e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=4}\log q_{1n}.K_1K_n\frac{d^2}{dv_1\,dv_n}}.\Theta_1.\Theta_3,$$

and obviously by transforming Θ_3 in the same manner we have

8. $$\Theta_4 = e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=4}\sum_{l=1}^{l=4}\log q_{nl} K_n K_l \frac{d^2}{dv_n dv_l}} \cdot \prod_{l=1}^{l=4} \Theta_{1_l}$$

where $$\prod_{l=1}^{l=4} \Theta_{1_l} = \Theta_1(v_1)\Theta_2(v_2)\Theta_3(v_3)\Theta_4(v_4),$$

and in the double summation n and l are always to have different values. The generalization of 8 for the p-tuple theta-functions is

9. $$\Theta_p = e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=p}\sum_{l=1}^{l=p}\log q_{nl} K_n K_l \frac{d^2}{dv_n dv_l}} \cdot \prod_{l=1}^{l=p} \Theta_{1_l}.$$

Equation 8 written out in full is

10. $$\vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_1 v_2 v_3 v_4)$$
$$= e^{-\frac{2}{\pi^2}\sum_{n=1}^{n=4}\sum_{l=1}^{l=4}\log q_{nl} K_n K_l \frac{d^2}{dv_n dv_l}} \cdot \vartheta\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}(v_1)\,\vartheta\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}(v_2)\,\vartheta\begin{pmatrix}\alpha_3\\ \beta_3\end{pmatrix}(v_3)\,\vartheta\begin{pmatrix}\alpha_4\\ \beta_4\end{pmatrix}(v_4).$$

The quantities K_1, K_2, K_3, K_4 may be taken as the complete elliptic integrals of the first kind corresponding to moduli k_1, k_2, k_3, k_4; similarly E_1, E_2, E_3, E_4 may be taken as the complete elliptic integrals of the second kind corresponding to the same moduli. Now writing $\vartheta = \vartheta\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(v)$ we have (Cayley's Elliptic Functions, page 227),

$$\frac{d^2\vartheta}{dv^2} - 2v\left(k'^2 - \frac{E}{K}\right)\frac{d\vartheta}{dv} + 2kk'^2\frac{d\vartheta}{dk} = 0,$$

or letting i denote either 1, 2, 3, or 4,

11. $$\frac{d^2\vartheta_i}{dv_i^2} - 2v_i\left(k_i'^2 - \frac{E_i}{K_i}\right)\frac{d\vartheta_i}{dv_i} + 2k_i k_i'^2\frac{d\vartheta_i}{dk_i} = 0.*$$

Now (Cayley, page 102) we have

12. $$\frac{dK_i}{dk_i} = \frac{1}{k_i k_i'^2}(E_i - k_i'^2 K_i),$$

so that 11 may be written in the form

13. $$\frac{d^2\vartheta_i}{dv_i^2} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d\vartheta_i}{dv_i} + 2k_i k_i'^2\frac{d\vartheta_i}{dk_i} = 0.$$

Differentiating this s times with respect to v_i and we have

14. $$\frac{d^2}{dv_i^2}\frac{d^s\vartheta_i}{dv_i^s} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d}{dv_i}\frac{d^s\vartheta_i}{dv_i^s} + 2k_i k_i'^2\frac{d}{dk_i}\frac{d^s\vartheta_i}{dv_i^s} = 0.$$

Now so far as x_i and k_i are concerned the general term in 8 is a numerical multiple of

$$K_i^s\frac{d^s\vartheta_i}{dv_i^s}, \text{ say } \phi_i,$$

then

15. $$\frac{d^2\varphi_i}{dv_i^2} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d\varphi_i}{dv_i} = K_i^s\left\{\frac{d^2}{dv_i^2}\frac{d^s\vartheta_i}{dv_i^s} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d\vartheta_i}{dv_i}\right\},$$

*For what immediately follows I am indebted to Mr. Forsyth's paper on Theta Functions: *Phil. Trans.*, 1882.

and

16. $$\frac{d\varphi_i}{dk_i} = K_i^s \left\{ \frac{d}{dk_i} \frac{d^s \vartheta_i}{dv_i^s} + \frac{s}{K_i} \frac{dK_i}{dk_i} \frac{d^s \vartheta_i}{dv_i^s} \right\}.$$

Hence

17. $$\frac{d^2 \varphi_i}{dv_i^2} + \frac{2k_i k_i'^2}{K_i} \frac{dK_i}{dk_i} v_i \frac{d\varphi_i}{dv_i} + 2k_i k_i'^2 \frac{d\varphi_i}{dk_i} = 0,$$

and now since Θ_4 is the sum of the quantities ϕ_i we have

18. $$\frac{d^2 \theta_4}{dv_i^2} + \frac{2k_i k_i'^2}{K_i} \frac{dK_i}{dk_i} v_i \frac{d\theta_4}{dv_i} + 2k_i k_i'^2 \frac{d\theta_4}{dk_i} = 0,$$

or

19. $$\frac{d^2 \theta_4}{dv_i^2} - 2v_i \left(k_i'^2 - \frac{E_i}{K_i} \right) \frac{d\theta_4}{dv_i} + 2k_i k_i'^2 \frac{d\theta_4}{dk_i} = 0.$$

There are of course four equations of this type. Now from 8 we have

20. $$\frac{d\theta_4}{dq_{nl}} = -\frac{2}{\pi^2} K_n K_l \frac{d^2}{dv_n dv_l} \cdot \frac{1}{q_{nl}} \cdot \Theta_4,$$

and consequently

21. $$q_{nl} \frac{d\theta_4}{dq_{nl}} + \frac{2K_n K_l}{\pi^2} \frac{d^2 \theta_4}{dv_n dv_l} = 0.$$

There are obviously six equations of this type. The quadruple theta-functions therefore satisfy in all ten different equations of the second order; written out in full these are

I
$$\begin{aligned}
&\frac{d^2 \theta_4}{dv_1^2} - 2v_1 \left(k_1'^2 - \frac{E_1}{K_1} \right) \frac{d\theta_4}{dv_1} + 2k_1 k_1'^2 \frac{d\theta_4}{dk_1} = 0 \\
&\frac{d^2 \theta_4}{dv_2^2} - 2v_2 \left(k_2'^2 - \frac{E_2}{K_2} \right) \frac{d\theta_4}{dv_2} + 2k_2 k_2'^2 \frac{d\theta_4}{dk_2} = 0 \\
&\frac{d^2 \theta_4}{dv_3^2} - 2v_3 \left(k_3'^2 - \frac{E_3}{K_3} \right) \frac{d\theta_4}{dv_3} + 2k_3 k_3'^2 \frac{d\theta_4}{dk_3} = 0 \\
&\frac{d^2 \theta_4}{dv_4^2} - 2v_4 \left(k_4'^2 - \frac{E_4}{K_4} \right) \frac{d\theta_4}{dv_4} + 2k_4 k_4'^2 \frac{d\theta_4}{dk_4} = 0.
\end{aligned}$$

II
$$\begin{aligned}
&q_{12} \frac{d\theta_4}{dq_{12}} + \frac{2K_1 K_2}{\pi^2} \frac{d^2 \theta_4}{dv_1 dv_2} = 0 \\
&q_{13} \frac{d\theta_4}{dq_{13}} + \frac{2K_1 K_3}{\pi^2} \frac{d^2 \theta_4}{dv_1 dv_3} = 0 \\
&q_{14} \frac{d\theta_4}{dq_{14}} + \frac{2K_1 K_4}{\pi^2} \frac{d^2 \theta_4}{dv_1 dv_4} = 0 \\
&q_{23} \frac{d\theta_4}{dq_{23}} + \frac{2K_2 K_3}{\pi^2} \frac{d^2 \theta_4}{dv_2 dv_3} = 0 \\
&q_{24} \frac{d\theta_4}{dq_{24}} + \frac{2K_2 K_4}{\pi^2} \frac{d^2 \theta_4}{dv_2 dv_4} = 0 \\
&q_{34} \frac{d\theta_4}{dq_{34}} + \frac{2K_3 K_4}{\pi^2} \frac{d^2 \theta_4}{dv_3 dv_4} = 0.
\end{aligned}$$

These equations can be obtained at once from the general definition of Θ_4; this is

22. $$\Theta_4, = \vartheta\begin{pmatrix}\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4\\ \beta_1\,\beta_2\,\beta_3\,\beta_4\end{pmatrix}(v_1 v_2 v_3 v_4), = \sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4}(-)^{\Sigma m\beta} q_1^{\frac{(2m_1+\alpha_1)^2}{4}} \dots q_4^{\frac{(2m_4+\alpha_4)^2}{4}}$$
$$\times\, q_{12}^{\frac{(2m_1+\alpha_1)(2m_2+\alpha_2)}{2}} \dots q_{34}^{\frac{(2m_3+\alpha_3)(2m_4+\alpha_4)}{2}}$$
$$\times\, e^{\frac{i\pi}{2}\left\{(2m_1+\alpha_1)\frac{v_1}{K_1}+\dots+(2m_4+\alpha_4)\frac{v_4}{K_4}\right\}}.$$

The general term in Θ_4 is a multiple of

$$q_i^{\frac{(2m_i+\alpha_i)^2}{4}} . e^{\frac{i\pi v}{2K_i}(2m_i+\alpha_i)} =, \text{ say } \mathfrak{B}_i,$$

the coefficient of $\mathfrak{B}_i$ being independent of v_i and k_i. Now

23. $$\frac{d}{dq_i}\mathfrak{B} = \frac{(2m_i+\alpha_i)^2}{4}\mathfrak{B}_i - \frac{i\pi v_i(2m_i+\alpha_i)}{2K_i^2}\frac{dK_i}{dp_i}\mathfrak{B}_i,$$

24. $$\frac{d^2}{dv_i^2}\mathfrak{B}_i = -\frac{(2m_i+\alpha_i)^2\pi^2}{4K_i^2}\mathfrak{B}_i,$$

25. $$v_i\frac{d}{dv_i}\mathfrak{B}_i = \frac{i\pi v_i(2m_i+\alpha_i)}{2K_i}\mathfrak{B}_i;$$

from these we derive

26. $$\frac{d}{dq_i}\mathfrak{B}_i = -\frac{K_i^2}{\pi^2 q_i}\frac{d^2}{dv_i^2}\mathfrak{B}_i - \frac{1}{K_i}\frac{dK_i}{dq_i}v_i\frac{d}{dv_i}\mathfrak{B}_i.$$

We have also

$$q_i = e^{-\pi\frac{K_i'}{K_i}},$$

and therefore

27. $$\frac{1}{q_i}\frac{dq_i}{dk_i} = -\pi\frac{K_i\frac{dK_i'}{dk_i} - K_i'\frac{dK_i}{dk_i}}{K_i^2}$$
$$= -\frac{\pi}{k_i k_i'^2 K_i^2}\{-K_i E_i' - K_i' E_i + K_i K_i'\}$$
$$= \frac{\pi^2}{2k_i k_i'^2 K_i^2},$$

since $K_i E_i' + K_i' E_i - K_i K_i' = \frac{\pi}{2}$. Multiply 26 by $\frac{dq_i}{dk_i}$ and substitute from 27 and we have $$\frac{d}{dk_i}\mathfrak{B}_i = -\frac{1}{2k_i k_i'^2}\frac{d^2}{dv_i^2}\mathfrak{B}_i - \frac{1}{K_i}\frac{dK_i}{dk_i}v_i\frac{d}{dv_i}\mathfrak{B}_i,$$

and hence Θ_4 which is the sum of the quantities $\mathfrak{B}_i$ satisfies the equation

$$\frac{d^2\Theta_4}{dv_i^2} + \frac{2k_i k_i'^2}{K_i}\frac{dK_i}{dk_i}v_i\frac{d\Theta_4}{dv_i} + 2k_i k_i'^2\frac{d\Theta_4}{dk_i} = 0,$$

or $$\frac{d^2\Theta_4}{dv_i^2} - 2v_i\left(k_i'^2 - \frac{E_i}{K_i}\right)\frac{d\Theta_4}{dv_i} + 2k_i k_i'^2\frac{d\Theta_4}{dk_i} = 0.$$

For the constants K, k we have of course the following relations:

$$K_1 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_1^2 \sin^2\varphi}}, \quad K_2 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_2^2 \sin^2\varphi}},$$

$$K_3 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_3^2 \sin^2\varphi}}, \quad K_4 = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k_4^2 \sin^2\varphi}},$$

or in the notation of hypergeometric series,

$$K_1 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_1^2), \quad K_2 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_2^2),$$

$$K_3 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_3^2), \quad K_4 = \frac{\pi}{2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k_4^2),$$

and for the constants E we have

$$E_1 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_1^2), \quad E_2 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_2^2),$$

$$E_3 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_3^2), \quad E_4 = \frac{\pi}{2} F(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k_4^2).$$

Also for the K's

$$\sqrt{\frac{2K_1}{\pi}} = 1 + 2q_1 + 2q_1^4 + 2q_1^9 + 2q_1^{16} + \dots$$

$$\sqrt{\frac{2K_2}{\pi}} = 1 + 2q_2 + 2q_2^4 + 2q_2^9 + 2q_2^{16} + \dots$$

$$\sqrt{\frac{2K_3}{\pi}} = 1 + 2q_3 + 2q_3^4 + 2q_3^9 + 2q_3^{16} + \dots$$

$$\sqrt{\frac{2K_4}{\pi}} = 1 + 2q_4 + 2q_4^4 + 2q_4^9 + 2q_4^{16} + \dots$$

and

$$k_1 = \left\{ \frac{2q_1^{\frac{1}{4}} + 2q_1^{\frac{9}{4}} + 2q_1^{\frac{25}{4}} + 2q_1^{\frac{49}{4}} + \dots}{1 + 2q_1 + 2q_1^4 + 2q_1^9 + 2q_1^{16} + \dots} \right\}^2$$

$$k_2 = \left\{ \frac{2q_2^{\frac{1}{4}} + 2q_2^{\frac{9}{4}} + 2q_2^{\frac{25}{4}} + 2q_2^{\frac{49}{4}} + \dots}{1 + 2q_2 + 2q_2^4 + 2q_2^9 + 2q_2^{16} + \dots} \right\}^2$$

$$k_3 = \left\{ \frac{2q_3^{\frac{1}{4}} + 2q_3^{\frac{9}{4}} + 2q_3^{\frac{25}{4}} + 2q_3^{\frac{49}{4}} + \dots}{1 + 2q_3 + 2q_3^4 + 2q_3^9 + 2q_3^{16} + \dots} \right\}^2$$

$$k_4 = \left\{ \frac{2q_4^{\frac{1}{4}} + 2q_4^{\frac{9}{4}} + 2q_4^{\frac{25}{4}} + 2q_4^{\frac{49}{4}} + \dots}{1 + 2q_4 + 2q_4^4 + 2q_4^9 + 2q_4^{16} + \dots} \right\}^2.$$

We have seen that it is possible to derive a quadruple theta-function by operating upon the product of a single and of a triple theta-function, and generally that a p-tuple theta-function is derived by performing a certain operation upon the product of a single and of a p-1-tuple theta-function; and finally that the

p-tuple function can be obtained by operating upon the product of p single theta-functions; *i. e.* calling the operator ∇ we have

$$\vartheta\binom{\alpha_1\,\alpha_2\ldots\alpha_p}{\beta_1\beta_2\ldots\beta_p}(v_1v_2\ldots v_p)=\nabla\cdot\vartheta\binom{\alpha_1}{\beta_1}(v_1)\,\vartheta\binom{\alpha_2}{\beta_2}(v_2)\ldots\vartheta\binom{\alpha_p}{\beta_p}(v_p).$$

It seems to me that it ought to be possible to obtain a theta-function of any order p by performing a proper operation upon certain combinations of theta-functions of *any* lower order. For example, cannot the quadruple function

$$\vartheta\binom{\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4}{\beta_1\beta_2\beta_3\beta_4}(v_1v_2v_3v_4)$$

be obtained by an operation performed upon a certain combination of the double theta-functions

$$\vartheta\binom{\alpha_1\alpha_2}{\beta_1\beta_2}(v_1,\,v_2),\quad\vartheta\binom{\alpha_1\alpha_3}{\beta_1\beta_3}(v_1,\,v_3),\quad\vartheta\binom{\alpha_1\alpha_4}{\beta_1\beta_4}(v_1,\,v_4),$$

$$\vartheta\binom{\alpha_2\alpha_3}{\beta_2\beta_3}(v_2,\,v_3),\quad\vartheta\binom{\alpha_2\alpha_4}{\beta_2\beta_4}(v_2,\,v_4),\quad\vartheta\binom{\alpha_3\alpha_4}{\beta_3\beta_4}(v_3,\,v_4)?$$

It is possible that some such expression may be known, but I have not seen it nor does it seem easy to obtain. It is, of course, perfectly simple to split up the right-hand members of equations 2 or 3 so that the double theta-function shall be brought in, but that reduction would obviously be of no value, as taking one further step we should arrive at an equation of the same form as 8. The question seems an interesting one and one worthy of investigation.

The theta-function under consideration may be written in the form

$$\vartheta\binom{\alpha_1\,\alpha_2\,\alpha_3\,\alpha_4}{\beta_1\beta_2\beta_3\beta_4}(v_1v_2v_3v_4)=\sum_{m_1}\sum_{m_2}\sum_{m_3}\sum_{m_4}(-)^{\Sigma m\beta}q_1^{\frac{(2m_1+\alpha_1)^2}{4}}\ldots q_4^{\frac{2(m_4+\alpha_4)^2}{4}}$$
$$\times\, q_{12}^{\frac{(2m_1+\alpha_1)(2m_2+\alpha_2)}{2}}\ldots q_{34}^{\frac{(2m_3+\alpha_3)(2m_4+\alpha_4)}{2}}$$
$$\times\, e^{\frac{i\pi}{2K_1}(2m_1+\alpha_1)v_1+\ldots\frac{i\pi}{2K_4}(2m_4+\alpha_4)v_4}.$$

The summations of course extend from $-\infty$ to $+\infty$ for all of the letters m_1, m_2, m_3, and m_4. This summation can be divided up into five parts: first, where all of the m's are zero, giving one term; second, where any three m's are zero while the fourth takes all values other than zero, giving four terms; third, where any two of the m's are zero and the other two take any values other than zero, giving six terms; fourth, where one of the m's is zero and the other three are not zero, giving four terms; and fifth, where none of the m's are zero, giving one term.

This gives us

$$\vartheta\binom{\alpha}{\beta}(v) = q_1^{\frac{\alpha_1^2}{4}} . q_2^{\frac{\alpha_2^2}{4}} . q_3^{\frac{\alpha_3^2}{4}} . q_4^{\frac{\alpha_4^2}{4}} . q_{12}^{\frac{\alpha_1 \alpha_2}{2}} \dots q_{34}^{\frac{\alpha_3 \alpha_4}{2}} . e^{\frac{i\pi}{2}\left[\frac{\alpha_1 v_1}{K_1} + \dots + \frac{\alpha_4 v_4}{K_4}\right]}$$

$$+ q_2^{\frac{\alpha_2^2}{4}} . q_3^{\frac{\alpha_3^2}{4}} . q_4^{\frac{\alpha_4^2}{4}} . q_{23}^{\frac{\alpha_2 \alpha_3}{2}} . q_{24}^{\frac{\alpha_2 \alpha_4}{2}} . q_{34}^{\frac{\alpha_3 \alpha_4}{2}} . e^{\frac{i\pi}{2}\left[\frac{\alpha_2 v_2}{K_2} + \dots + \frac{\alpha_4 v_4}{K_4}\right]}$$

$$\times \sum_{m_1=-\infty}^{m_1=+\infty} (-)^{m_1 \beta_1} q_1^{\frac{(2m_1+\alpha_1)^2}{4}} q_{12}^{\frac{\alpha_2(2m_1+\alpha_1)}{2}} \dots q_{14}^{\frac{\alpha_4(2m_1+\alpha_1)}{2}} . e^{\frac{i\pi}{2K_1}(2m_1+\alpha_1)v_1}$$

$+$ (three similar terms)

$$+ q_3^{\frac{\alpha_3^2}{4}} . q_4^{\frac{\alpha_4^2}{4}} . q_{34}^{\frac{\alpha_3 \alpha_4}{2}} . e^{\frac{i\pi}{2}\left[\frac{\alpha_3 v_3}{K_3} + \frac{\alpha_4 v_4}{K_4}\right]}$$

$$\times \sum_{m_1=-\infty}^{m_1=\infty} \sum_{m_2=-\infty}^{m_2=\infty} (-)^{m_1\beta_1 + m_2\beta_2} . q_1^{\frac{(2m_1+\alpha_1)^2}{4}} . q_2^{\frac{2(m_2+\alpha_2)^2}{4}} \dots q_{12}^{\frac{(2m_1+\alpha_1)(2m_2+\alpha_2)}{2}} . q_{13}^{\frac{\alpha_3(2m_1+\alpha_1)}{2}}$$

$$\times q_{14}^{\frac{\alpha_4(2m_1+\alpha_1)}{2}} . q_{23}^{\frac{\alpha_3(2m_2+\alpha_2)}{2}} . q_{24}^{\frac{\alpha_4(2m_2+\alpha_2)}{2}} . e^{\frac{i\pi}{2}\left[\frac{(2m_1+\alpha_1)v_1}{K_1} + \frac{(2m_2+\alpha_2)v_2}{K_2}\right]}$$

$+$ (five similar terms) $+$ etc.

The remaining terms are formed in the manner indicated above, but it is not worth while writing them down, as the formula is too complicated to deal with in the general case. There is one class of cases, or rather one group of functions, sixteen in number, for which this formula may be very much simplified—these are the functions for which all of the indices α are zero. The five sets of terms in the general formula correspond to the following values of the α's.

	α_1	α_2	α_3	α_4
I	0	0	0	0
II	1	0	0	0
	0	1	0	0
	0	0	1	0
	0	0	0	1
III	1	1	0	0
	0	1	1	0
	0	0	1	1
	1	0	0	1
	1	0	1	0
	0	1	0	1
IV	1	1	1	0
	0	1	1	1
	1	0	1	1
	1	1	0	1
V	1	1	1	1

Combining each of these 16 groups of the values of $\alpha_1 \ldots \alpha_4$ with the corresponding 16 groups for the values of $\beta_1 \ldots \beta_4$ we have the whole 256 functions. The simplest way of representing each of these functions by a formula similar to the general one given above would be to work it out *ab initio*, assuming the particular characteristic and developing for that special case, but a table of that magnitude could hardly have any practical value. We may take, however, the case where

$$(\alpha_1 \alpha_2 \alpha_3 \alpha_4) = (0\,0\,0\,0)$$

and give the β-row all of its values—we thus have the sixteen functions referred to above. Substitute these values of the α's in the general formula and make also

$$(\beta_1 \beta_2 \beta_3 \beta_4) = (0\,0\,0\,0)$$

then we have

$$\vartheta\begin{pmatrix}0\,0\,0\,0\\0\,0\,0\,0\end{pmatrix}(v_1,\, v_2,\, v_3,\, v_4) =, \text{ say } \vartheta_0(v) =$$

$$1 + 2\sum_{m_1=1}^{m_1=\infty} q_1^{m_1^2} \cos\frac{m_1 \pi v_1}{K_1} + \ldots + 2\sum_{m_4=1}^{m_4=\infty} q_4^{m_4^2} \cos\frac{m_4 \pi v_4}{K_4}$$

$$+\sum_{m_1=-\infty}^{m_1=+\infty}\ \sum_{m_2=-\infty}^{m_2=+\infty} q_1^{m_1^2} q_2^{m_2^2} q_{12}^{2m_1 m_2} e^{i\pi\left[\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2}\right]} + \text{(five similar terms)}$$

$$+\sum_{m_1=-\infty}^{m_1=+\infty}\ \sum_{m_2=-\infty}^{m_2=+\infty}\ \sum_{m_3=-\infty}^{m_3=+\infty} q_1^{m_1^2} q_2^{m_2^2} q_3^{m_3^2} q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3} e^{i\pi\left[\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3}\right]}$$

$$+ \text{(three similar terms)}$$

$$+\sum_{m_1=-\infty}^{m_1=+\infty}\ \sum_{m_2=-\infty}^{m_2=+\infty}\ \sum_{m_3=-\infty}^{m_3=+\infty}\ \sum_{m_4=-\infty}^{m_4=+\infty} q_1^{m_1^2} \ldots q_4^{m_4^2} . q_{12}^{2m_1 m_2} \ldots q_{34}^{2m_3 m_4} e^{i\pi\left[\frac{m_1 v_1}{K_1} + \ldots \frac{m_4 v_4}{K_4}\right]}.$$

In this last term the values $m_1,\ m_2,\ m_3,\ m_4 = 0$ are excluded.

The first line in the right-hand member of this equation is, in the ordinary notation,

$$\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_3) + \vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_4) - 3.$$

The first term in the group of double summations is the double theta-function

$$\vartheta\begin{pmatrix}0\,0\\0\,0\end{pmatrix}(v_1 v_2).$$

Transforming this in the same manner we have (see Forsyth's memoir, page 809)

$$\vartheta\begin{pmatrix}0\,0\\0\,0\end{pmatrix}(v_1 v_2) = 1 + 2\sum_{m_1=1}^{m_1=\infty} q_1^{m_1^2} \cos\frac{m_1 \pi v_1}{K_1} + 2\sum_{m_2=1}^{m_2=\infty} q_2^{m_2^2} \cos\frac{m_2 \pi v_2}{K_2}$$

$$+ 2\sum_{m_1=1}^{m_1=\infty}\ \sum_{m_2=1}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2} \left\{ q_{12}^{2m_1 m_2} \cos\pi\left(\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2}\right) + q_{12}^{-2m_1 m_2} \cos\pi\left(\frac{m_1 v_1}{K_1} - \frac{m_2 v_2}{K_2}\right)\right\}$$

or
$$\vartheta\begin{pmatrix}0\,0\\0\,0\end{pmatrix}(v_1v_2)=\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1)+\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2)-1$$
$$+2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty}q_1^{m_1^2}q_2^{m_2^2}\left\{q_{12}^{2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}\right)+q_{12}^{-2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}\right)\right\}.$$

The remaining five terms under the double summation are at once obtained from this.

Take now the terms under the triple summation sign. The first of these is
$$\vartheta\begin{pmatrix}0\,0\,0\\0\,0\,0\end{pmatrix}(v_1v_2v_3).$$
We reduce this just as we reduced the quadruple function, and we have
$$\vartheta\begin{pmatrix}0\,0\,0\\0\,0\,0\end{pmatrix}(v_1v_2v_3)=1+2\sum_{m_1=1}^{m_1=\infty}q_1^{m_1^2}\cos\frac{m_1\pi v_1}{K_1}+\ldots+2\sum_{m_3=1}^{m_3=\infty}q_3^{m_3^2}\cos\frac{m_3\pi v_3}{K_3}$$
$$+\sum_{m_1=-\infty}^{m_1=\infty}\sum_{m_2=-\infty}^{m_2=\infty}q_1^{m_1^2}q_2^{m_2^2}q_{12}^{2m_1m_2}e^{i\pi\left[\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}\right]}+\text{(two similar terms)}$$
$$+\sum_{m_1=-\infty}^{m_1=\infty}\sum_{m_2=-\infty}^{m_2=\infty}\sum_{m_3=-\infty}^{m_3=\infty}q_1^{m_1^2}q_2^{m_2^2}q_3^{m_3^2}q_{12}^{2m_1m_2}q_{13}^{2m_1m_3}q_{23}^{2m_2m_3}e^{i\pi\left[\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_4}\right]}.$$

In this last term the values m_1, m_2, $m_3=0$ are excluded.

The first five terms on the right-hand side of this equation are
$$=\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1)+\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2)+\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_3)-2.$$
The terms under the double summation sign are
$$\vartheta\begin{pmatrix}0\,0\\0\,0\end{pmatrix}(v_1v_2)=\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1)+\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2)-1$$
$$+2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty}q_1^{m_1^2}q_2^{m_2^2}\left\{q_{12}^{2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}\right)+q_{12}^{-2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}\right)\right\}$$
and two similar ones. Collecting together all our results we have now
$$\vartheta\begin{pmatrix}0000\\0000\end{pmatrix}(v_1v_2v_3v_4)=7\left\{\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_1)+\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_2)+\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_3)\right\}+4\vartheta\begin{pmatrix}0\\0\end{pmatrix}(v_4)-14+\mathfrak{A}.$$
Here
$$\mathfrak{A}=2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty}q_1^{m_1^2}q_2^{m_2^2}\left\{q_{12}^{2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}\right)+q_{12}^{-2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}\right)\right\}$$
$$+\text{(five similar terms)}$$

$$+ 2\sum_{m_1=1}^{m_1=\infty}\sum_{m_2=1}^{m_2=\infty} q_1^{m_1^2} q_2^{m_2^2}\left\{ q_{12}^{2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}+\frac{m_2 v_2}{K_2}\right) + q_{12}^{-2m_1 m_2}\cos\pi\left(\frac{m_1 v_1}{K_1}-\frac{m_2 v_2}{K_2}\right)\right\}$$

$+$ (eleven similar terms)

$$+\sideset{}{'}\sum_{m_1=-\infty}^{\infty}\sideset{}{'}\sum_{m_2=-\infty}^{\infty}\sideset{}{'}\sum_{m_3=-\infty}^{\infty} q_1^{m_1^2}\ldots q_3^{m_3^2} q_{12}^{2m_1 m_2}\ldots q_{23}^{2m_2 m_3} e^{i\pi\left[\frac{m_1 v_1}{K_1}+\ldots+\frac{m_3 v_3}{K_4}\right]}$$

$$+\sideset{}{'}\sum_{m_1=-\infty}^{\infty}\sideset{}{'}\sum_{m_2=-\infty}^{\infty}\sideset{}{'}\sum_{m_3=-\infty}^{\infty}\sideset{}{'}\sum_{m_4=-\infty}^{\infty} q_1^{m_1^2}\ldots q_4^{m_4^2} q_{12}^{2m_1 m_2}\ldots q_{34}^{2m_3 m_4} e^{i\pi\left[\frac{m_1 v_1}{K_1}+\ldots+\frac{m_4 v_4}{K_4}\right]}.$$

The accented Σ means that the values $m=0$ are to be left out in forming the sum.

We proceed now to a more complete reduction of the quantity $\mathfrak{A}$. The terms under the double summation sign require only to be added together, no further reduction being necessary. The terms under each of the triple summation signs divide up into four groups, viz., taking the first case where the summations refer to m_1, m_2 and m_3 we have

I.	m_1, m_2, m_3 all positive	One term.
II.	Any two of these positive and the third negative .	Three terms.
III.	Any one positive and the remaining two negative .	Three terms.
IV.	All three negative	One term.
	In all	Eight terms.

For the quadruple summations we have similarly sixteen terms. Taking now the case of the triple summations and we find the following values, in which for convenience I have written

$$Q_4 = q_1^{m_1^2} q_2^{m_2^2} q_3^{m_3^2}, \quad u = \frac{v}{K}.$$

I. m_1, m_2, m_3 all positive.

$$\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3} . e^{i\pi(m_1 u_1 + m_2 u_2 + m_3 u_3)}.$$

II. Any two positive and the third negative.

$$\left\{\begin{array}{l}
\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{2m_2 m_3} . e^{i\pi(-m_1 u_1 + m_2 u_2 + m_3 u_3)} \\
\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{-2m_2 m_3} . e^{i\pi(m_1 u_1 - m_2 u_2 + m_3 u_3)} \\
\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{-2m_2 m_3} . e^{i\pi(m_1 u_1 + m_2 u_2 - m_3 u_3)}.
\end{array}\right.$$

III. Any two negative and the third positive.

$$\begin{cases} \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{-2m_2 m_3} . e^{i\pi(-m_1 u_1 - m_2 u_2 + m_3 u_3)} \\ \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{-2m_2 m_3} . e^{i\pi(-m_1 u_1 + m_2 u_2 - m_3 u_3)} \\ \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{2m_2 m_3} . e^{i\pi(m_1 u_1 - m_2 u_2 - m_3 u_3)} . \end{cases}$$

IV. All three negative.

$$\sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3} . e^{-i\pi(m_1 u_1 + m_2 u_2 + m_3 u_3)} .$$

The summations are of course for m_1, m_2 and m_3.

Adding groups I and IV and we have obviously

$$2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3} \cos \pi \left(\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3} \right).$$

The first of group II added to the third of group III gives

$$2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{2m_2 m_3} \cos \pi \left(-\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3} \right).$$

The second of II added to the second of III gives

$$2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{-2m_2 m_3} \cos \pi \left(\frac{m_1 v_1}{K_1} - \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3} \right).$$

The third of II added to the first of III gives

$$2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{-2m_2 m_3} \cos \pi \left(\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} - \frac{m_3 v_3}{K_3} \right).$$

Combining all of these we have for the first term under the triple summation sign

$$\begin{aligned} & 2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{2m_2 m_3} \cos \pi \left(\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3} \right) \\ + & 2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{2m_2 m_3} \cos \pi \left(-\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3} \right) \\ + & 2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{-2m_1 m_2} q_{13}^{2m_1 m_3} q_{23}^{-2m_2 m_3} \cos \pi \left(\frac{m_1 v_1}{K_1} - \frac{m_2 v_2}{K_2} + \frac{m_3 v_3}{K_3} \right) \\ + & 2 \sum_1^\infty \sum_1^\infty \sum_1^\infty Q_4 q_{12}^{2m_1 m_2} q_{13}^{-2m_1 m_3} q_{23}^{-2m_2 m_3} \cos \pi \left(\frac{m_1 v_1}{K_1} + \frac{m_2 v_2}{K_2} - \frac{m_3 v_3}{K_3} \right). \end{aligned}$$

The remaining three terms are of course obtained by replacing the suffixes (1, 2, 3) by (2, 3, 4), (3, 4, 1) and (4, 1, 2).

Take up now the quadruple summation. As already remarked there are, in this case, sixteen groups of terms, these combine however into eight. The process of reduction is exactly the same as in the case of the triple summation,

so it is not necessary to go further into it. We have, in fact, for the quadruple summation the following value, where for brevity I have written

$$Q = q_1^{m_1^2} q_2^{m_2^2} q_3^{m_3^2} q_4^{m_4^2},$$

$$2\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty Q q_{12}^{2m_1m_2} \ldots q_{34}^{2m_3m_4} \cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+\,2\sum\sum\sum\sum Q q_{12}^{-2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{2m_2m_3}q_{24}^{2m_2m_4}q_{34}^{2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+\,2\sum\sum\sum\sum Q q_{12}^{-2m_1m_2}q_{13}^{2m_1m_3}q_{14}^{2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{2m_3m_4}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+\,2\sum\sum\sum\sum Q q_{12}^{2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+\,2\sum\sum\sum\sum Q q_{12}^{2m_1m_2}q_{13}^{2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)$$

$$+\,2\sum\sum\sum\sum Q q_{12}^{2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+\,2\sum\sum\sum\sum Q q_{12}^{-2m_1m_2}q_{13}^{2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+\,2\sum\sum\sum\sum Q q_{12}^{-2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{2m_1m_4}q_{23}^{2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)$$

The limits of the summations in the last seven terms are of course the same as in the first term. We have now merely to add together all of our results in order to obtain the value of the quantity $\mathfrak{A}$. Notice that in the above given value of $\mathfrak{A}$ of the "*(eleven similar terms)*" one is identical in form and value with the one written down, and the remaining ten terms reduce to five each of which is repeated once. We have then finally

$$\vartheta\binom{0000}{0000}(v_1v_2v_3v_4) = 7\left\{\vartheta\binom{0}{0}(v_1)+\vartheta\binom{0}{0}(v_2)+\vartheta\binom{0}{0}(v_3)\right\}+4\vartheta\binom{0}{0}(v_4)-14$$

$$+\,4\sum_1^\infty\sum_1^\infty q_1^{m_1^2}q_2^{m_2^2}\left\{q_{12}^{2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}\right)+q^{-2m_1m_2}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}\right)\right\}$$

+ (five similar terms)

$$+\,2\sum_1^\infty\sum_1^\infty\sum_1^\infty Q_4\left\{q_{12}^{2m_1m_2}q_{13}^{2m_1m_3}q_{23}^{2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)\right.$$

$$+\,q_{12}^{-2m_1m_2}q_{13}^{-2m_1m_3}q_{23}^{2m_2m_3}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)$$

$$+\,q_{12}^{-2m_1m_2}q_{13}^{2m_1m_3}q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}\right)$$

$$\left.+\,q_{12}^{2m_1m_2}q_{13}^{-2m_1m_3}q_{23}^{-2m_2m_3}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}\right)\right\}$$

+ (three similar terms)

$$+2\sum_1^\infty\sum_1^\infty\sum_1^\infty\sum_1^\infty Q\left\{q_{12}^{2m_1m_2}q_{13}^{2m_1m_3}q_{14}^{2m_1m_4}q_{23}^{2m_2m_3}q_{24}^{2m_2m_4}q_{34}^{2m_3m_4}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)\right.$$

$$+q_{12}^{-2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{2m_2m_3}q_{24}^{2m_2m_4}q_{34}^{2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+q_{12}^{-2m_1m_2}q_{13}^{2m_1m_3}q_{14}^{2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{2m_3m_4}\cos\pi\left(\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+q_{12}^{2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+q_{12}^{2m_1m_2}q_{13}^{2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)$$

$$+q_{12}^{2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}-\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$+q_{12}^{-2m_1m_2}q_{13}^{2m_1m_3}q_{14}^{-2m_1m_4}q_{23}^{-2m_2m_3}q_{24}^{2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}-\frac{m_3v_3}{K_3}+\frac{m_4v_4}{K_4}\right)$$

$$\left.+q_{12}^{-2m_1m_2}q_{13}^{-2m_1m_3}q_{14}^{2m_1m_4}q_{23}^{2m_2m_3}q_{24}^{-2m_2m_4}q_{34}^{-2m_3m_4}\cos\pi\left(-\frac{m_1v_1}{K_1}+\frac{m_2v_2}{K_2}+\frac{m_3v_3}{K_3}-\frac{m_4v_4}{K_4}\right)\right\}.$$

In this formula and in those preceding it is of course not strictly accurate to write the quadruple function in the form

$$\vartheta\begin{pmatrix}0000\\0000\end{pmatrix}(v_1v_2v_3v_4);$$

it really should be written

$$\vartheta\begin{pmatrix}0000\\0000\end{pmatrix}(u_1u_2u_3u_4),$$

but as the quantities v_1, v_2, v_3, v_4 appear in the right-hand members of the equations I have written them in the left-hand members also. The equations connecting the v's and the u's are as above

$$K_1u_1=v_1,\quad K_2u_2=v_2,\quad K_3u_3=v_3,\quad K_4u_4=v_4.$$

The remaining 255 quadruple theta-functions may be thrown into forms similar to the one just examined, but as the formulæ are so very long and as the process has been already sufficiently indicated, it is not worth while to work out any more of them.

It would not be difficult to extend the above processes to the case of p-tuple theta-functions, but it is not the object of the present paper to do so. Still it may be remarked that the term in the p-tuple theta-function corresponding to the term $\mathfrak{A}$ just computed will consist of a group of terms involving double summations, another group involving triple summations, and so on until we come to a single term which involves a p-tuple summation. The method of computing

these terms is identical with that above given for the quadruple functions. Take for example $r < p$, then the r-tuple summations are divided up as follows: suppose first r even, say $r = 2\rho$.

		Number of Cases.
$(1)_1$	All the m's positive, ...	1
$(1)_2$	All the m's but one positive, .	2ρ
$(1)_3$	All the m's but two positive, .	$\frac{2\rho(2\rho-1)}{2}$
. . . .	. . . .	. . . .
$(1)_{\rho+1}$	All the m's but ρ positive, . .	$\frac{2\rho(2\rho-1)(2\rho-2)\ldots(\rho+1)}{\rho}$
. . . .	. . . .	. . . .
$(1)_{2\rho-1}$	All the m's negative but two .	$\frac{2\rho(2\rho-1)}{2}$
$(1)_{2\rho}$	All the m's negative but one .	2ρ
$(1)_{2\rho+1}$	All the m's negative, ...	1

Here the first and last terms are to be combined, also the second and the last but one, and so on until we come to the middle term; this term is made up of

$$\frac{2\rho(2\rho-1)(2\rho-2)\ldots\rho+1}{\rho}$$

(an even number) of simple terms, and the first half of these simple terms is to be combined with the second half in order to obtain forms similar to those given in the case of the quadruple theta-functions. The case where r is odd follows at once from the case of r even.

On the Absolute Classification of Quadratic Loci, and on their Intersections with each other and with Linear Loci.

By William E. Story.

By the absolute classification of loci I mean that classification which is not altered by any real linear transformation, and which is identical with the ordinary classification in so far as the latter is independent of all consideration of the nature of the infinite elements of the loci. Many of the results here obtained are well known, but I believe some of them are new, and the collection of the criteria in such a form that they can be applied to any real forms of the equations of quadratic and linear loci will probably not be without a practical interest. The geometrical conditions are combinations of contact and real and imaginary intersection; the corresponding algebraical criteria can be put into a form in which certain invariants or combinations of the coefficients of covariants are distinguished as zero or positive or negative. A part of this classification has been made in essence by Professor Sylvester in a communication to the Philosophical Magazine, in February, 1851, namely that part which relates to the *contacts of two loci* of the second order; but the form of the investigation is necessarily quite different from that here adopted, where a distinction has to be made between cases in which the relations as to reality as well as contact are different, and where the conditions are required explicitly in terms of the coefficients of the equations of the loci. Homogeneous coordinates are employed throughout the paper; it is assumed that real elements (points, lines and planes) have real coordinates, and that the equations of the loci have real coefficients. Each locus is considered as belonging to a space of the smallest number of dimensions in which it can exist.

I. The equation of a pair of points on a given straight line is

$$U \equiv ax^2 + 2hxy + by^2 = 0,$$

and the discriminant of U is $\Delta = ab - h^2 = (ab)$, say;

if $(ab) > 0$, the points are conjugate imaginary,

if $(ab) < 0$, the points are real and distinct,

if $(ab) = 0$, the points are real and coincident.

II. The equations of a conic and a straight line in a given plane are

$$U \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$
$$P \equiv \alpha x + \beta y + \gamma z = 0.$$

The intersections of U and P satisfy the equation

$$(a\gamma^2 - 2g\alpha\gamma + c\alpha^2)x^2 + 2(h\gamma^2 - f\alpha\gamma - g\beta\gamma + c\alpha\beta)xy + (b\gamma^2 - 2f\beta\gamma + c\beta^2)y^2 = 0,$$

found by eliminating z between $U = 0$ and $P = 0$, the discriminant of which is

$$\begin{vmatrix} a\gamma^2 - 2g\alpha\gamma + c\alpha^2 & , & h\gamma^2 - f\alpha\gamma - g\beta\gamma + c\alpha\beta \\ h\gamma^2 - f\alpha\gamma - g\beta\gamma + c\alpha\beta, & & b\gamma^2 - 2f\beta\gamma + c\beta^2 \end{vmatrix} = -\gamma^2 \begin{vmatrix} a & h & g & \alpha \\ h & b & f & \beta \\ g & f & c & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix}$$

$= -\gamma^2 \Delta_P = -\gamma^2 (abc)_P$, say; hence, by I,

if $(abc)_P < 0$, U and P intersect in imaginary points,

if $(abc)_P > 0$, U and P intersect in distinct real points,

if $(abc)_P = 0$, U and P intersect in coincident real points.

If $\gamma = 0$ the above discriminant vanishes identically, but eliminating x or y instead of z, the discriminant of the resultant equation is $-\alpha^2\Delta_P$ or $-\beta^2\Delta_P$, and the criteria are the same as those just given.

$\Delta_P = 0$ is in general the condition that P is tangent to U (*i. e.* is the tangential equation of U), but if U consists of two straight lines, $\Delta_P = 0$ is the condition that P passes through the double point of U.

III. The nature of the conic U, where

$$U \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

is found by considering its intersections with a real line rotating about a fixed point, namely: if the intersections are imaginary for all positions of the rotating line, U is an imaginary true conic; if the intersections are imaginary for all positions but one, U consists of a pair of conjugate imaginary straight lines whose intersection lies on the rotating line in that one position; if the intersections are real for an infinite succession of positions, and imaginary for an infinite succession, U is a real true conic; and if the intersections are real for all positions, U is either a real true conic or consists of a pair of real or imaginary straight lines.

The case in which U consists of two straight lines meeting in the fixed point of the rotating line is the only case in which this method does not fully determine the nature of the curve, and this uncertainty as to the reality of the lines in this case is removed by considering the intersections with *two* lines rotating about different points. Of course the pair of lines is distinguished from the true conic by the vanishing of the discriminant. For convenience I consider the intersections of U with *three* lines rotating about the vertices of the triangle of reference.

Let

$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, let $A = (bc)$, $B = (ac)$, $C = (ab)$, F, G, H be the first minors of Δ,

and let $z = \rho x$ be one of the rotating lines; then the intersections of U with this line for any given value of ρ satisfy the equation

$$(a + 2g\rho + c\rho^2)x^2 + 2(h + f\rho)xy + by^2 = 0,$$

the discriminant of which is

$$Q \equiv b(a + 2g\rho + c\rho^2) - (h + f\rho)^2 = C - 2G\rho + A\rho^2.$$

If $\Delta \lessgtr 0$, U is a true conic:

if $A \lessgtr 0$, $AQ = (A\rho - G)^2 + AC - G^2 = (A\rho - G)^2 + b\Delta$, and Q will be positive for every real value of ρ if $b\Delta > 0$, and $A > 0$, and only then;

if $A = 0$, $Q = C - 2G\rho$, and Q will be of the same sign for every real value of ρ if $G = 0$, but then $b\Delta = AC - G^2 = 0$, *i. e.* $b = 0$, and hence $C = ab - h^2 = -h^2$, *i. e.* $C \overline{\leq} 0$, hence Q cannot be positive for every real value of ρ.

Hence U is an imaginary true conic if $b\Delta > 0$ and $A > 0$, and a real true conic if $\Delta \lessgtr 0$ and either $b\Delta \overline{\leq} 0$ or $A \overline{\leq} 0$. If $b\Delta > 0$ and $A > 0$, then evidently also $a\Delta > 0$, $c\Delta > 0$, $B > 0$, $C > 0$; and the conditions $b\Delta > 0$ and $A > 0$ may be replaced by $a\Delta > 0$ and $C > 0$.

If $\Delta = 0$, U consists of a pair of straight lines:

if $A \lessgtr 0$, $AQ = (A\rho - G)^2 > 0$, hence $Q > 0$ for all real values of ρ but one, and the lines of U are imaginary, if $A > 0$;

if $A = 0$, but either $B \lessgtr 0$ or $C \lessgtr 0$, U will consist of imaginary lines if those of the quantities B, C which do not vanish are positive;

if $A = B = C = 0$, *i. e.* $bc - f^2 = 0$, $ac - g^2 = 0$, $ab - h^2 = 0$; if $a \lessgtr 0$, $c = \frac{g^2}{a}$, $b = \frac{h^2}{a}$, $f^2 = \frac{g^2h^2}{a^2}$, $f = \pm\frac{gh}{a}$, then $\Delta = 2\frac{g^2h^2}{a}(\pm 1 - 1)$, *i. e.* if $g \lessgtr 0$ and

$h \lessgtr 0$ the upper sign must be given to f, and $U \equiv (ax + by + gz)^2$, *i. e.* U consists of two coincident real lines; if $a \lessgtr 0$ and $h = 0$, then $b = f = 0$, and $U \equiv ax^2 + 2gzx + cz^2$, where $ac - g^2 = 0$, *i. e.* U consists of two coincident real lines; similarly if $a \lessgtr 0$ and $g = 0$; if $a = 0$, and either $b \lessgtr 0$ or $c \lessgtr 0$, then $g = h = 0$, $U \equiv by^2 + 2fyz + cz^2$, where $bc - f^2 = 0$, *i. e.* U still consists of two coincident real lines; if $a = b = c = 0$, then $f = g = h = 0$ and there is no conic.

The criteria for the conic U are then the following:

$(abc) \lessgtr 0$, $a(abc) > 0$, $(ab) > 0$, U is an imaginary true conic;

$(abc) \lessgtr 0$, and either $a(abc) \overline{\lessdot} 0$ or $(ab) \overline{\lessdot} 0$, U is a real true conic;

$(abc) = 0$, and either $(bc) > 0$ or $(ac) > 0$ or $(ab) > 0$, U is a pair of imaginary lines;

$(abc) = 0$, and either $(bc) < 0$ or $(ac) < 0$ or $(ab) < 0$, U is a pair of distinct real lines;

$(abc) = 0$, and $(bc) = (ac) = (ab) = 0$, U is a pair of coincident real lines.

IV. If U and V are two conics, where

$$U \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$
$$V \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy,$$

among the conics $\lambda U + \mu V = 0$ passing through the intersections of U and V are three pairs of lines for which $\lambda : \mu$ is determined by the equation $\Delta_{\lambda\mu} = 0$, where

$$\Delta_{\lambda\mu} \equiv \begin{vmatrix} \lambda a + \mu a', & \lambda h + \mu h', & \lambda g + \mu g' \\ \lambda h + \mu h', & \lambda b + \mu b', & \lambda f + \mu f' \\ \lambda g + \mu g', & \lambda f + \mu f', & \lambda c + \mu c' \end{vmatrix};$$

if the intersections of U, V are all real, the three pairs of lines are real; if two of the intersections of U, V are real and two imaginary, the lines of one pair are real and those of each of the two other pairs are non-conjugate imaginary; if the intersections of U, V are all imaginary, the lines of one pair are real and those of each of the other pairs are conjugate imaginary. Hence if the four intersections of U, V are all real or all imaginary the three roots of $\Delta_{\lambda\mu} = 0$ are real, but if U and V have two real and two imaginary intersections, one of the roots of $\Delta_{\lambda\mu} = 0$ is real and two are (of course conjugate) imaginary. Putting then*

$$R = \Theta^2\Theta'^2 + 18\Delta\Delta'\Theta\Theta' - 27\Delta^2\Delta'^2 - 4\Delta\Theta'^3 - 4\Delta'\Theta^3,$$

* Salmon's Conic Sections, § 372.

$\Delta_{\lambda\mu} = 0$ will have three distinct real roots if $R > 0$, two imaginary roots and one real root if $R < 0$, and three real roots of which two at least are equal if $R = 0$.

If $R < 0$, U and V have two distinct real intersections and two imaginary intersections.

If $R > 0$, U and V have four distinct real intersections or four distinct imaginary intersections, and it remains to distinguish between these two cases.* For then the roots of $\Delta_{\lambda\mu} = 0$ are all real and distinct, and hence the reduction to the common polar triangle is real, *i. e.* by a real linear transformation U and V can be brought at once to the forms

$$U_0 \equiv \alpha x_0^2 + \beta y_0^2 + \gamma z_0^2, \quad V_0 \equiv \alpha' x_0^2 + \beta' y_0^2 + \gamma' z_0^2.$$

Now the intersections will be all imaginary if an imaginary conic whose equation is real can be found passing through these intersections, *i. e.* if $\lambda U + \mu V = 0$ represents an imaginary conic for any real values of λ, μ; by the criteria of III this will happen for any values of λ, μ, if $\lambda\alpha + \mu\alpha'$, $\lambda\beta + \mu\beta'$, $\lambda\gamma + \mu\gamma'$ have the same sign, *i. e.* if λ, μ, ρ, p, q, r can be determined as real quantities such that

$$\lambda\alpha + \mu\alpha' = \rho p^2, \quad \lambda\beta + \mu\beta' = \rho q^2, \quad \lambda\gamma + \mu\gamma' = \rho r^2,$$

i. e. if real quantities p, q, r can be so determined that

$$\begin{vmatrix} \alpha & \alpha' & p^2 \\ \beta & \beta' & q^2 \\ \gamma & \gamma' & r^2 \end{vmatrix} = (\beta\gamma' - \beta'\gamma)\,p^2 + (\gamma\alpha' - \gamma'\alpha)\,q^2 + (\alpha\beta' - \alpha'\beta)\,r^2 = 0,$$

which will be possible if $(\beta\gamma' - \beta'\gamma)$, $(\gamma\alpha' - \gamma'\alpha)$, $(\alpha\beta' - \alpha'\beta)$ are not all of the same sign, and only then, *i. e.* U and V will have four real intersections if $(\beta\gamma' - \beta'\gamma)$, $(\gamma\alpha' - \gamma'\alpha)$, $(\alpha\beta' - \alpha'\beta)$ are all of the same sign. Now let $\mathbf{F}$ be the quadratic covariant of U and V, *i. e.*†

$$\begin{aligned} \mathbf{F} \equiv (BC' + B'C - 2FF')\,x^2 + (CA' + C'A - 2GG')\,y^2 + (AB' + A'B - 2HH')\,z^2 \\ + 2(GH' + G'H - AF' - A'F)\,yz + 2(HF' + H'F - BG' - B'G)\,zx \\ + 2(FG' + F'G - CH' - C'H)\,xy, \end{aligned}$$

where A, B, C, F, G, H and A', B', C', F', G', H' are the first minors of Δ and Δ', the discriminants of U and V respectively. This covariant formed for U_0 and V_0 is

$$\mathbf{F}_0 \equiv \alpha\alpha'(\beta\gamma' + \gamma\beta')\,x_0^2 + \beta\beta'(\gamma\alpha' + \alpha\gamma')\,y_0^2 + \gamma\gamma'(\alpha\beta' + \beta\alpha')\,z_0^2;$$

the roots of $\Delta_{\lambda\mu} = 0$ formed for U, V, and for U_0, V_0 are evidently the same, and if the coefficients α, β, γ, α', β', γ' are so taken in absolute value that the two

* Salmon says they "have not been distinguished by any simple criterion."

† Salmon's Conic Sections, § 378.

equations $\Delta_{\lambda\mu} = 0$ are not simply equivalent but identical, *i. e.* if the determinant of substitution is unity, then, as in Salmon's Geometry of Three Dimensions, §§ 215, 234,

$$\Delta = \alpha\beta\gamma, \ \Delta' = \alpha'\beta'\gamma', \ U \equiv U_0, \ V \equiv V_0, \ \mathbf{F} \equiv \mathbf{F}_0,$$

which gives the three equation for x_0^2, y_0^2, z_0^2,

$$\alpha x_0^2 + \beta y_0^2 + \gamma z_0^2 \equiv U,$$
$$\alpha' x_0^2 + \beta' y_0^2 + \gamma' z_0^2 \equiv V,$$
$$\alpha\alpha'(\beta\gamma' + \gamma\beta')x_0^2 + \beta\beta'(\gamma\alpha' + \alpha\gamma')y_0^2 + \gamma\gamma'(\alpha\beta' + \beta\alpha')z_0^2 \equiv \mathbf{F},$$

from which follow

$$x_0^2 : y_0^2 : z_0^2 \equiv \begin{vmatrix} U, & \beta, & \gamma \\ V, & \beta', & \gamma' \\ \mathbf{F}, & \beta\beta'(\gamma\alpha' + \alpha\gamma'), & \gamma\gamma'(\alpha\beta' + \beta\alpha') \end{vmatrix} : \text{etc.}$$
$$\equiv (\beta\gamma' - \gamma\beta')(-\alpha\beta'\gamma' U - \alpha'\beta\gamma V + \mathbf{F}) : \text{etc.}$$
$$\equiv (\beta\gamma' - \gamma\beta')\left(-\frac{\alpha}{\alpha'}\Delta' U - \frac{\alpha'}{\alpha}\Delta V + \mathbf{F}\right) : \text{etc.}$$

or, introducing the roots of $\Delta_{\lambda\mu} = 0$, $\frac{\lambda_1}{\mu_1} = -\frac{\alpha'}{\alpha}$, $\frac{\lambda_2}{\mu_2} = -\frac{\beta'}{\beta}$, $\frac{\lambda_3}{\mu_3} = -\frac{\gamma'}{\gamma}$, and putting

$$\frac{\mu_1}{\lambda_1}\Delta' U + \frac{\lambda_1}{\mu_1}\Delta V + \mathbf{F} \equiv X, \ \frac{\mu_2}{\lambda_2}\Delta' U + \frac{\lambda_2}{\mu_2}\Delta V + \mathbf{F} \equiv Y, \ \frac{\mu_3}{\lambda_3}\Delta' U + \frac{\lambda_3}{\mu_3}\Delta V + \mathbf{F} \equiv Z,$$

we have $\qquad x_0^2 : y_0^2 : z_0^2 \equiv (\beta\gamma' - \gamma\beta')X : (\gamma\alpha' - \alpha\gamma')Y : (\alpha\beta' - \beta\alpha')Z,$

from which it is evident that $(\beta\gamma' - \gamma\beta')$, $(\gamma\alpha' - \alpha\gamma')$, $(\alpha\beta' - \beta\alpha')$ have respectively the signs of X, Y, Z, or their opposites, *i. e.* the intersections of U, V will be all four real if the three last mentioned quantities have the same signs, and only then. But X, Y, Z are real, since the roots of $\Delta_{\lambda\mu} = 0$ are real, hence they will have the same sign if their sum s_1 and product s_3 have the same sign and the sum of their binary products s_2 is positive.

Let σ_1 be the sum, σ_3 the product, σ_2 the sum of binary products of $\frac{\lambda_1}{\mu_1}$, $\frac{\lambda_2}{\mu_2}$, $\frac{\lambda_3}{\mu_3}$, then we have, writing $\Delta_{\lambda\mu} \equiv \Delta.\lambda^3 + \Theta.\lambda^2\mu + \Theta'.\lambda\mu^2 + \Delta'.\mu^3$,

$$\Delta.\sigma_1 = -\Theta, \ \Delta.\sigma_2 = \Theta', \ \Delta.\sigma_3 = -\Delta',$$

$$s_1 = \frac{\sigma_2}{\sigma_3}\Delta'.U + \sigma_1\Delta.V + 3\mathbf{F} = -\Theta'.U - \Theta.V + 3\mathbf{F},$$

$$s_2 = \frac{\sigma_1}{\sigma_3}\Delta'^2.U^2 + \frac{1}{3}\frac{\sigma_1\sigma_2 - 3\sigma_3}{\sigma_3}\Delta\Delta'.UV + \sigma_2\Delta^2.V^2 + 2\frac{\sigma_2}{\sigma_3}\Delta'.U\mathbf{F}$$
$$+ 2\sigma_1\Delta.V\mathbf{F} + 3\mathbf{F}^2$$
$$= \Delta'\Theta.U^2 + \tfrac{1}{3}(\Theta\Theta' - 3\Delta\Delta')UV + \Delta\Theta'.V^2 - 2\Theta'.U\mathbf{F} - 2\Theta.V\mathbf{F} + 3\mathbf{F}^2,$$

$$s_3 = \frac{1}{\sigma_3}\Delta'^3 . U^3 + \frac{\sigma_1^2 - 2\sigma_2}{\sigma_3}\Delta\Delta'^2 . U^2 V + \frac{\sigma_2^2 - 2\sigma_1\sigma_3}{\sigma_3}\Delta^2\Delta' . UV^2 + \sigma_3\Delta^3 . V^3$$
$$+ \frac{\sigma_1}{\sigma_3}\Delta'^2 . U^2\mathbf{F} + \frac{1}{3}\frac{\sigma_1\sigma_2 - 3\sigma_3}{\sigma_3}\Delta\Delta' . UV\mathbf{F} + \sigma_2\Delta^2 . V^2\mathbf{F} + \frac{\sigma_2}{\sigma_3}\Delta' . U\mathbf{F}^2$$
$$+ \sigma_1\Delta . V\mathbf{F}^2 + \mathbf{F}^3 = -\Delta\Delta'^2 . U^3 + (2\Delta\Theta' - \Theta^2)\Delta' . U^2 V$$
$$+ (2\Delta' . \Theta - \Theta'^2)\Delta . UV^2 - \Delta^2\Delta' . V^3 + \Delta'\Theta . U^2\mathbf{F} + \tfrac{1}{3}(\Theta\Theta' - 3\Delta\Delta') . UV\mathbf{F}$$
$$+ \Delta\Theta' . V^2\mathbf{F} - \Theta' . U\mathbf{F}^2 - \Theta . V\mathbf{F}^2 + \mathbf{F}^3,$$

where*

$$\Theta = Aa' + Bb' + Cc' + 2Ff' + 2Gg' + 2Hh',$$
$$\Theta' = A'a + B'b + C'c + 2F'f + 2G'g + 2H'h.$$

If then $s_1 s_3 \geqq 0$ and $s_2 > 0$ for all values of the variables, U and V have four distinct real intersections, otherwise they have four distinct imaginary intersections. It may be remarked that we cannot have $s_1 = 0$ if $s_2 > 0$, and $s_3 = 0$ only for those values of the variables for which $x_0 = 0$, $y_0 = 0$, or $z_0 = 0$, *i. e.* for points on the sides of the common self-conjugate triangle.

If $R = 0$, U and V have contact, *i. e.* two of their intersections are coincident. If the other two are distinct from each other and from the double intersection, this is real and the distinct intersections may be real or conjugate imaginary; in the former case the three pairs of lines are real and two of them coincide, in the latter case the lines of one pair are real and those of the other pairs (which coincide) conjugate imaginary. In either case $\Delta_{\lambda\mu} = 0$ has three real roots, two of which are coincident. These cases may be distinguished by a method similar to that employed above for $R > 0$, namely, U and V cannot be put into the canonical form before used, but they can be brought by a real linear transformation into the forms†

$$U_0 \equiv \alpha x_0^2 + \beta y_0^2 + 2\gamma z_0 x_0, \quad V_0 \equiv \alpha' x_0^2 + \beta' y_0^2 + 2\gamma' z_0 x_0 .$$

The equation $\Delta_{\lambda\mu} = 0$ formed for U_0, V_0 is

$$0 = \begin{vmatrix} \lambda\alpha + \mu\alpha', & 0 & , \lambda\gamma + \mu\gamma' \\ 0 & , \lambda\beta + \mu\beta', & 0 \\ \lambda\gamma + \mu\gamma', & 0 & , \quad 0 \end{vmatrix} = -(\lambda\beta + \mu\beta')(\lambda\gamma + \mu\gamma')^2,$$

i. e. if $\frac{\lambda_1}{\mu_1}$ is the double and $\frac{\lambda_2}{\mu_2}$ the single root, $\frac{\lambda_1}{\mu_1} = -\frac{\gamma'}{\gamma}$, $\frac{\lambda_2}{\mu_2} = -\frac{\beta'}{\beta}$.

The pairs of straight lines through the intersections are then

$$(\gamma\alpha' - \alpha\gamma')x_0^2 - (\beta\gamma' - \gamma\beta')y_0^2 = 0 \text{ and } (\alpha\beta' - \beta\alpha')x_0^2 - 2(\beta\gamma' - \gamma\beta')z_0 x_0 = 0,$$

and the former is the double pair. This double pair will be real or imaginary

* Salmon's Conic Sections, § 370. † Clebsch's Vorlesungen über Geometrie, I, p. 137.

according as $\beta\gamma' - \gamma\beta'$ and $\gamma\alpha' - \alpha\gamma'$ are of like or unlike signs, *i. e.* the distinct intersections of U, V will be real or imaginary in these cases respectively. Now the covariant $\mathbf{F}$ formed for U_0, V_0 is

$$\mathbf{F}_0 \equiv -(\alpha\beta\gamma'^2 + \gamma^2\alpha'\beta')x_0^2 - 2\beta\gamma\beta'\gamma'y_0^2 - 2\gamma\gamma'(\beta\gamma' + \gamma\beta')z_0x_0,$$

and, if the determinant of substitution for passing from U, V to U_0, V_0 is unity,

$$\Delta = -\beta\gamma^2, \quad \Delta' = -\beta'\gamma'^2,$$
$$\alpha x_0^2 + \beta y_0^2 + 2\gamma z_0 x_0 \equiv U,$$
$$\alpha' x_0^2 + \beta' y_0^2 + 2\gamma' z_0 x_0 \equiv V,$$
$$-(\alpha\beta\gamma'^2 + \gamma^2\alpha'\beta')x_0^2 - 2\beta\gamma\beta'\gamma'y_0^2 - 2\gamma\gamma'(\beta\gamma' + \gamma\beta')z_0x_0 \equiv \mathbf{F},$$

from which follows $\left(\text{observing that here } \frac{\lambda_3}{\mu_3} = \frac{\lambda_1}{\mu_1}\right)$,

$$X \equiv Z \equiv \frac{\mu_1}{\lambda_1}\Delta'.U + \frac{\lambda_1}{\mu_1}\Delta.V + \mathbf{F} \equiv (\beta\gamma' - \gamma\beta')(\gamma\alpha' - \alpha\gamma')x_0^2,$$
$$Y \equiv \frac{\mu_2}{\lambda_2}\Delta'.U + \frac{\lambda_2}{\mu_2}\Delta.V + \mathbf{F} \equiv (\beta\gamma' - \gamma\beta')^2 y_0^2;$$

hence $Y \overline{\underset{>}{}} 0$ and $X \equiv Z \overline{\underset{>}{}} 0$ or $\overline{\underset{<}{}} 0$ according as $(\beta\gamma' - \gamma\beta')$ and $(\gamma\alpha' - \alpha\gamma')$ have like or unlike signs, *i. e.* according as the distinct intersections of U and V are real or imaginary; also, using s_1, s_2, s_3 as above,

$$s_1 \equiv (\beta\gamma' - \gamma\beta')[2(\gamma\alpha' - \alpha\gamma')x_0^2 + (\beta\gamma' - \gamma\beta')y_0^2],$$
$$s_2 \equiv (\beta\gamma' - \gamma\beta')^2(\gamma\alpha' - \alpha\gamma')[(\gamma\alpha' - \alpha\gamma')x_0^2 + 2(\beta\gamma' - \gamma\beta')y_0^2]x_0^2,$$
$$s_3 \equiv (\beta\gamma' - \gamma\beta')^4(\gamma\alpha' - \alpha\gamma')^2 x_0^4 y_0^2,$$

i. e. $s_1 \overline{\underset{>}{}} 0$, $s_2 \overline{\underset{>}{}} 0$, $s_3 \overline{\underset{>}{}} 0$ for all real values of the variables if the distinct intersections of U and V are real, and only then ($s_1 = 0$ only for the point of contact, $s_2 = 0$ only for a point on the double common tangent, and $s_3 = 0$ only for a point on the double common tangent or on the common polar with respect to U and V of the intersection of that tangent with the common chord joining the distinct intersections); *i. e.* (noticing that $s_3 \overline{\underset{>}{}} 0$ for all real values of the variables) if any set of real values of the variables exists for which either $s_1 < 0$ or $s_2 < 0$, the distinct intersections of U and V are imaginary.

If U and V have two distinct pairs of coincident intersections, *i. e.* if they have two distinct contacts, we may employ the canonical forms*

$$U \equiv \beta y_0^2 + 2\alpha z_0 x_0, \quad V \equiv \beta' y_0^2 + 2\alpha' z_0 x_0;$$

then $\Delta = -\alpha^2\beta$, $\Delta' = -\alpha'^2\beta'$, $\Delta_{\lambda\mu} \equiv -(\lambda\alpha + \mu\alpha')^2(\lambda\beta + \mu\beta')$, $R = 0$,

$$\frac{\lambda_1}{\mu_1} = \frac{\lambda_3}{\mu_3} = -\frac{\alpha'}{\alpha}, \quad \frac{\lambda_2}{\mu_2} = -\frac{\beta'}{\beta}, \quad \alpha\beta' - \beta\alpha' \gtrless 0,$$

* Clebsch, Vorlesungen über Geometrie, I, p. 139.

and $\alpha' U - \alpha V \equiv \lambda_1 U + \mu_1 V = 0$ represents a double straight line (the chord of contact), while $\beta' U - \beta V \equiv \lambda_2 U + \mu_2 V = 0$ represents the tangents at the contacts, *i. e.* a pair of distinct straight lines, real or imaginary according as the two contacts are real or imaginary. Now in this case

$$\mathbf{F} \equiv -2\alpha\alpha' [\beta\beta' y_0^2 + (\alpha\beta' + \beta\alpha') z_0 x_0] \equiv -\alpha\alpha' (\beta' U + \beta V),$$

and, using $X \equiv Z$ and Y as before,

$$X \equiv \frac{\mu_1}{\lambda_1} \Delta'.U + \frac{\lambda_1}{\mu_1} \Delta.V + \mathbf{F} \equiv \alpha\alpha' (\beta' U + \beta V) - \alpha\alpha' (\beta' U + \beta V) \equiv 0,$$

$$Y \equiv \frac{\mu_2}{\lambda_2} \Delta'.U + \frac{\lambda_2}{\mu_2} \Delta.V + \mathbf{F} \equiv \beta\alpha'^2.U + \alpha^2\beta'.V - \alpha\alpha' (\beta' U + \beta V)$$

$$\equiv (\alpha\beta' - \beta\alpha')(\alpha' U - \alpha V) \equiv -(\alpha\beta' - \beta\alpha')^2 y_0^2,$$

$$s_1 \equiv 2X + Y \equiv Y, \quad s_2 \equiv X^2 + 2XY \equiv 0, \quad s_3 \equiv X^2 Y \equiv 0;$$

so that for all values of the variables $s_1 \equiv Y \leqq 0$. The double root of

$$\Delta_{\lambda\mu} \equiv \Delta.\lambda^3 + \Theta.\lambda^2\mu + \Theta'.\lambda\mu^2 + \Delta'.\mu^3 = 0$$

is given by
$$\frac{\lambda_1}{\mu_1} = -\frac{2(3\varDelta'\theta - \theta'^2)}{9\varDelta\varDelta' - \theta\theta'} = -\frac{9\varDelta\varDelta' - \theta\theta'}{2(3\varDelta\theta' - \theta^2)},$$

so that
$$\frac{\lambda_1^2}{\mu_1^2} = \frac{3\varDelta'\theta - \theta'^2}{3\varDelta\theta' - \theta^2},$$

and the distinct root is given by

$$\frac{\lambda_2}{\mu_2} = -\frac{\varDelta'\mu_1^2}{\varDelta\lambda_1^2} = -\frac{\varDelta'(3\varDelta\theta' - \theta^2)}{\varDelta(3\varDelta'\theta - \theta'^2)},$$

and therefore the contacts will be real or imaginary according as

$$\Delta'(3\Delta\Theta' - \Theta^2) U - \Delta(3\Delta'\Theta - \Theta'^2) V = 0$$

represents a pair of real or imaginary straight lines, *i. e.*, by the criteria obtained in III, according as those of the three quantities

$$L = \varDelta'^2(3\varDelta\theta' - \theta^2)^2 A - \varDelta\varDelta'(3\varDelta\theta' - \theta^2)(3\varDelta'\theta - \theta'^2)(bc' + cb' - 2ff') + \varDelta^2(3\varDelta'\theta - \theta'^2)^2 A',$$
$$M = \varDelta'^2(3\varDelta\theta' - \theta^2)^2 B - \varDelta\varDelta'(3\varDelta\theta' - \theta^2)(3\varDelta'\theta - \theta'^2)(ca' + ac' - 2gg') + \varDelta^2(3\varDelta'\theta - \theta'^2)^2 B',$$
$$N = \varDelta'^2(3\varDelta\theta' - \theta^2)^2 C - \varDelta\varDelta'(3\varDelta\theta' - \theta^2)(3\varDelta'\theta - \theta'^2)(ab' + ba' - 2hh') + \varDelta^2(3\varDelta'\theta - \theta'^2)^2 C'$$

which do not vanish are negative or positive (namely, we have proved above that such of them as do not vanish are of the same sign, and evidently they will all vanish only when the tangents at the contacts coincide, *i. e.* when U and V have four-point contact).

If three of the intersections of U and V coincide, *i. e.* if U and V have three-point contact, the three roots of $\Delta_{\lambda\mu} = 0$ are equal, *i. e.*

$$\frac{3\varDelta'}{\theta'} = \frac{\theta'}{\theta} = \frac{\theta}{3\varDelta} = \sqrt[3]{\frac{\varDelta'}{\varDelta}} = -\frac{\lambda_1}{\mu_1} = -\frac{\lambda_2}{\mu_2} = -\frac{\lambda_3}{\mu_3}$$

(and of course $R=0$), and $\lambda_1 U+\mu_1 V=0$, *i. e.* $\sqrt[3]{\Delta'}U-\sqrt[3]{\Delta}V=0$, where the real cube roots are to be taken, represents a pair of lines consisting of the common tangent at the triple intersection and the line joining this point with the isolated intersection. The equation of these lines may be written in either of the forms $3\Delta' U-\Theta' V=0$, $\Theta' U-\Theta V=0$ or $\Theta U-3\Delta V=0$, but either of these forms becomes illusory in some case (namely, the first two if $\Delta'=0$, and the last two if $\Delta=0$, for then $\Theta=0$ and $\Theta'=0$); but the form above used is always definite. In this case the canonical forms

$$U\equiv\beta y_0^2+2\gamma z_0 x_0,\quad V\equiv\rho\beta y_0^2+2\rho\gamma z_0 x_0+2\alpha' x_0 y_0$$

may be employed; then

$$\Delta=-\beta\gamma^2,\ \Delta'\equiv-\rho^3\beta\gamma^2,\ \Delta_{\lambda\mu}\equiv-\beta\gamma^2(\lambda+\mu\rho)^3,\ \frac{\lambda_1}{\mu_1}=-\rho,$$
$$\mathbf{F}\equiv\gamma^2[\alpha'^2x_0^2-2\rho^2\beta^2y_0^2-4\rho^2\beta\gamma z_0x_0-2\rho\beta\alpha' x_0y_0]$$
$$\equiv\gamma^2[\alpha'^2x_0^2-\rho\beta(\rho U+V)],$$
$$X\equiv Y\equiv Z\equiv\rho\beta\gamma^2(\rho U+V)+\mathbf{F}\equiv\gamma^2\alpha'^2x_0^2,$$
$$s_1\equiv3\gamma^2\alpha'^2x_0^2,\ s_2\equiv3\gamma^4\alpha'^4x_0^4,\ s_3\equiv\gamma^6\alpha'^6x_0^6,$$

i. e. $s_1 \overset{=}{>} 0$, $s_2 \overset{=}{>} 0$, $s_3 \overset{=}{>} 0$.

If all four intersections of U and V coincide, *i. e.* if U and V have four-point contact, the three roots of $\Delta_{\lambda\mu}=0$ are equal, *i. e.*

$$\frac{3\Delta'}{\Theta'}=\frac{\Theta'}{\Theta}=\frac{\Theta}{3\Delta}=\sqrt[3]{\frac{\Delta'}{\Delta}}=-\frac{\lambda_1}{\mu_1}=-\frac{\lambda_2}{\mu_2}=-\frac{\lambda_3}{\mu_3}$$

(and of course $R=0$), and $\sqrt[3]{\Delta'}U-\sqrt[3]{\Delta}V=0$ represents double the tangent at the quadruple intersection, *i. e.* by III,

$$\sqrt[3]{\Delta'^2}A-\sqrt[3]{\Delta\Delta'}(bc'+cb'-2ff')+\sqrt[3]{\Delta^2}A'=0,$$
$$\sqrt[3]{\Delta'^2}B-\sqrt[3]{\Delta\Delta'}(ca'+ac'-2gg')+\sqrt[3]{\Delta^2}B'=0,$$
$$\sqrt[3]{\Delta'^2}C-\sqrt[3]{\Delta\Delta'}(ab'+ba'-2hh')+\sqrt[3]{\Delta^2}C'=0.$$

Convenient canonical forms of U and V are*

$$U\equiv\alpha x_0^2+\beta y_0^2+2\gamma x_0y_0+2\delta z_0x_0,\quad V\equiv\rho\alpha x_0^2+\sigma(\beta y_0^2+2\gamma x_0y_0+2\delta z_0x_0);$$

then

$$\Delta=-\beta\delta^2,\ \Delta'=-\sigma^3\beta\delta^2,\ \Delta_{\lambda\mu}\equiv-\beta\delta^2(\lambda+\mu\sigma)^3,\ \frac{\lambda_1}{\mu_1}=-\sigma,$$
$$\mathbf{F}\equiv-\sigma\beta\delta^2[(\rho+\sigma)\alpha x_0^2+2\sigma\beta y_0^2+4\sigma\delta z_0x_0+4\sigma\gamma x_0y_0]$$
$$\equiv-\sigma\beta\delta^2(\sigma U+V),$$
$$X\equiv Y\equiv Z\equiv\sigma\beta\delta^2(\sigma U+V)+\mathbf{F}\equiv0,$$
$$s_1\equiv0,\ s_2\equiv0,\ s_3\equiv0.$$

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* Clebsch, Geometrie, p. 141.

If U and V have a straight line in common, the conic $\lambda U + \mu V$ contains this line, whatever the values of λ, μ; hence $\Delta_{\lambda\mu} \equiv 0$, *i. e.*

$$\Delta = 0, \quad \Theta = 0, \quad \Theta' = 0, \quad \Delta' = 0.$$

If the second lines of U and V are distinct from each other and from the first line, and the double points are also distinct, we may write

$$U \equiv 2\beta x_0 y_0, \quad V \equiv 2\gamma' x_0 z_0, \quad \mathbf{F} \equiv \beta^2 \gamma'^2 x_0^2,$$

i. e. $\mathbf{F} \overset{=}{>} 0$ for all real values of the variables.

If the second lines of U and V are distinct from each other and from the first line, but the double points coincident, we may write

$$U \equiv 2\beta x_0 y_0, \quad V \equiv a' x_0^2 + 2\beta' x_0 y_0, \quad \mathbf{F} \equiv 0,$$

but, since the lines of each pair are distinct and real, either $A < 0$ or $B < 0$ or $C < 0$, and either $A' < 0$ or $B' < 0$ or $C' < 0$.

If the second line of U coincides with the line common to U and V, we may write $\qquad U \equiv a x_0^2, \quad V \equiv 2\beta' x_0 y_0, \quad \mathbf{F} \equiv 0,$

and then $\qquad A = B = C = 0$, and either $A' < 0$ or $B' < 0$ or $C' < 0$.

Finally, if U and V are identical,

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}.$$

Collecting these results, the criteria for the intersections of the two conics U and V are as follows:

1. $R < 0$; two real and distinct and two conjugate imaginary intersections.
2. $R > 0$, $s_2 > 0$, $s_1 s_3 \overset{=}{>} 0$; four distinct real intersections.
3. $R > 0$, but not at once, $s_2 > 0$ and $s_1 s_3 \overset{=}{>} 0$; four distinct imaginary intersections.

$$R = 0, \text{ but not } \frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta}.$$

4. $s_1 \overset{=}{>} 0$, $s_2 \overset{=}{>} 0\, [s_3 \overset{=}{>} 0]$; two coincident real and two distinct real intersections.
5. Not at once $s_1 \overset{=}{>} 0$ and $s_2 \overset{=}{>} 0$, nor $s_2 \equiv 0$, $[s_3 \overset{=}{>} 0]$; two coincident real and two distinct imaginary intersections.
6. $s_1 \overset{=}{<} 0$, $s_2 \equiv 0\, [s_3 \equiv 0]$, and either $L < 0$ or $M < 0$ or $N < 0$; two distinct pairs of coincident real intersections.
7. $s_1 \overset{=}{<} 0$, $s_2 \equiv 0\, [s_3 \equiv 0]$, and either $L > 0$ or $M > 0$ or $N > 0$; two distinct conjugate pairs of coincident imaginary intersections.

$$\frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta}, \text{ but not } \Delta = \Theta = \Theta' = \Delta' = 0.$$

8. $s_1 \overset{=}{>} 0\, [s_2 \overset{=}{>} 0,\ s_3 \overset{=}{>} 0]$; three coincident and one isolated intersections, all real.
9. $s_1 \equiv 0\, [s_2 \equiv 0,\ s_3 \equiv 0]$; four coincident real intersections.

$$\Delta = \Theta = \Theta' = \Delta' = 0, \text{ but not } \frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}.$$

10. $\mathbf{F} \underset{>}{=} 0$; straight line common, second line of each pair distinct from the other and from the common line, double points distinct.

11. $\mathbf{F} \equiv 0$, either $A < 0$ or $B < 0$ or $C < 0$, either $A' < 0$ or $B' < 0$ or $C' < 0$; straight line common, second line of each pair distinct from the other and from the common line, double points coincident.

12. $\left.\begin{array}{l} \mathbf{F} \equiv 0,\ A = B = C = 0, \text{ either } A' < 0 \text{ or } B' < 0 \text{ or } C' < 0; \\ \text{or } \mathbf{F} \equiv 0,\ A' = B' = C' = 0, \text{ either } A < 0 \text{ or } B < 0 \text{ or } C < 0; \end{array}\right\}$

straight line common which is a double line of one pair and single line of the other pair.

13. $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}$; conics coincident.

The conditions enclosed in brackets are superfluous.

The cases for which $R = 0$ but not $\frac{3\Delta'}{\theta'} = \frac{\theta'}{\theta} = \frac{\theta}{3\Delta}$ can be distinguished from each other also without the use of s_1, s_2, s_3. In these cases $\Delta_{\lambda\mu} = 0$ has a double root $\frac{\lambda_1}{\mu_1}$ and a single root $\frac{\lambda_2}{\mu_2}$. For these roots we have

$$\frac{\lambda_1}{\mu_1} = -\frac{2(3\Delta'\theta - \theta'^2)}{9\Delta\Delta' - \theta\theta'} = -\frac{9\Delta\Delta' - \theta\theta'}{2(3\Delta\theta' - \theta^2)}, \quad \frac{\lambda_1^2}{\mu_1^2} = \frac{3\Delta'\theta - \theta'^2}{3\Delta\theta' - \theta^2},$$

$$\frac{\lambda_2}{\mu_2} = -\frac{\Delta'}{\Delta}\frac{\mu_1^2}{\lambda_1^2} = -\frac{\Delta'(3\Delta\theta' - \theta^2)}{\Delta(3\Delta'\theta - \theta'^2)},$$

and $\lambda_1 U + \mu_1 V = 0$ may be written

$$(9\Delta\Delta' - \Theta\Theta')U - 2(3\Delta\Theta' - \Theta^2)V = 0 \text{ or } 2(3\Delta'\Theta - \Theta'^2)U - (9\Delta\Delta' - \Theta\Theta')V = 0,$$

the nature of which is determined by the principal first minors of its determinant, which are, by virtue of the equation $R = 0$,

$$2(3\Delta\Theta' - \Theta^2)I,\ 2(3\Delta\Theta' - \Theta^2)J,\ 2(3\Delta\Theta' - \Theta^2)K$$

or

$$2(3\Delta'\Theta - \Theta'^2)I,\ 2(3\Delta'\Theta - \Theta'^2)J,\ 2(3\Delta'\Theta - \Theta'^2)K,$$

where

$$I = 2(3\Delta'\Theta - \Theta'^2)A - (9\Delta\Delta' - \Theta\Theta')(bc' + cb' - 2ff') + 2(3\Delta\Theta' - \Theta^2)A',$$

$$J = 2(3\Delta'\Theta - \Theta'^2)B - (9\Delta\Delta' - \Theta\Theta')(ca' + ac' - 2ff') + 2(3\Delta\Theta' - \Theta^2)B',$$

$$K = 2(3\Delta'\Theta - \Theta'^2)C - (9\Delta\Delta' - \Theta\Theta')(ab' + ba' - 2hh') + 2(3\Delta\Theta' - \Theta^2)C';$$

namely, the first form of the principal minors must be employed if $\Delta' = \Theta' = 0$, the second if $\Delta = \Theta = 0$, but otherwise either.

The criteria for cases 4–7 may then be stated thus:

4. Some one of the quantities I, J, K has the same sign as $3\Delta\Theta' - \Theta^2$ or $3\Delta'\Theta - \Theta'^2$.

5 Some one of the quantities I, J, K has a different sign from $3\Delta\Theta' - \Theta^2$ or $3\Delta'\Theta - \Theta'^2$.

6. $I = J = K = 0$ and either $L < 0$ or $M < 0$ or $N < 0$.

7. $I = J = K = 0$ and either $L > 0$ or $M > 0$ or $N > 0$.

In all of these criteria a vanishing quantity is regarded as having no sign.

The cases in which $\frac{3\Delta'}{\Theta'} = \frac{\Theta'}{\Theta} = \frac{\Theta}{3\Delta}$ but not $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \frac{f'}{f} = \frac{g'}{g} = \frac{h'}{h}$ may also be distinguished thus:

8. $\mathbf{F}$ not $\equiv 0$; 9. $\mathbf{F} \equiv 0$:

Or again by the nature of $\sqrt[3]{\Delta'}U - \sqrt[3]{\Delta}V = 0$, *i. e.* whether it consists of two distinct or two coincident lines, *i. e.* if

$$I' = \sqrt[3]{\Delta'^2}A - 2\sqrt[3]{\Delta\Delta'}(bc' + cb' - 2ff') + \sqrt[3]{\Delta^2}A',$$
$$J' = \sqrt[3]{\Delta'^2}B - 2\sqrt[3]{\Delta\Delta'}(ca' + ac' - 2gg') + \sqrt[3]{\Delta^2}B',$$
$$K' = \sqrt[3]{\Delta'^2}C - 2\sqrt[3]{\Delta\Delta'}(ab' + ba' - 2hh') + \sqrt[3]{\Delta^2}C',$$

then the criteria are:

8. $I' < 0$ or $J' < 0$ or $K' < 0$;

9. $I' = J' = K' = 0$.

V. The nature of the intersection of a quadric surface U and a straight line given as the intersections of two planes P and P', where

$$U \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw,$$
$$P \equiv \alpha x + \beta y + \gamma z + \delta w, \quad P' \equiv \alpha' x + \beta' y + \gamma' z + \delta' z,$$

is determined by the nature of the roots of the quadratic obtained by eliminating two of the variables from the equations $U = 0$, $P = 0$, $P' = 0$, but a more symmetrical method may be employed. Let x_1, y_1, z_1, w_1 and x_2, y_2, z_2, w_2 be the coordinates of any two different points on the line $P = 0$, $P' = 0$, then the coordinates of any point of this line will be

$$\sigma x_1 + \tau x_2, \quad \sigma y_1 + \tau y_2, \quad \sigma z_1 + \tau z_2, \quad \sigma w_1 + \tau w_2,$$

and for the intersections of the line with the quadric,

$$\sigma^2 U_{11} + 2\sigma\tau U_{12} + \tau^2 U_{22} = 0,$$

where U_{11} and U_{22} are the results of substituting in U the coordinates x_1, y_1, z_1, w_1 and x_2, y_2, z_2, w_2 respectively, and

$$U_{12} \equiv \frac{1}{2}\left(x_1 \frac{\partial U_2}{\partial x_2} + y_1 \frac{\partial U_2}{\partial y_2} + z_1 \frac{\partial U_2}{\partial z_2} + w_1 \frac{\partial U_2}{\partial w_2}\right)$$
$$\equiv \frac{1}{2}\left(x_2 \frac{\partial U_1}{\partial x_1} + y_2 \frac{\partial U_1}{\partial y_1} + z_2 \frac{\partial U_1}{\partial z_1} + w_2 \frac{\partial U_1}{\partial w_1}\right).$$

The nature of the intersections, as is evident from I., depends upon the value of $U_{11}U_{22} - U_{12}^2$, which contains the coordinates only in determinants of the second order; writing for convenience

$$y_1z_2 - y_2z_1 = (y_1z_2), \text{ etc.,} \quad \beta\gamma' - \beta'\gamma = (\beta\gamma'), \text{ etc.,}$$

we have*

$$\frac{(y_1z_2)}{(\alpha\delta')} = \frac{(z_1x_2)}{(\beta\delta')} = \frac{(x_1y_2)}{(\gamma\delta')} = \frac{(x_1w_2)}{(\beta\gamma')} = \frac{(y_1w_2)}{(\gamma\alpha')} = \frac{(z_1w_2)}{(\alpha\beta')}$$

$= \rho$, say, where ρ is a non-evanescent real quantity. Now

$$U_{11}U_{22} - U_{12}^2 = \tfrac{1}{4}\begin{vmatrix} x_1\frac{\partial U_1}{\partial x_1} + y_1\frac{\partial U_1}{\partial y_1} + z_1\frac{\partial U_1}{\partial z_1} + w_1\frac{\partial U_1}{\partial w_1}, & x_2\frac{\partial U_1}{\partial x_1} + y_2\frac{\partial U_1}{\partial y_1} + z_2\frac{\partial U_1}{\partial z_1} + w_2\frac{\partial U_1}{\partial w_1} \\ x_1\frac{\partial U_2}{\partial x_2} + y_1\frac{\partial U_2}{\partial y_2} + z_1\frac{\partial U_2}{\partial z_2} + w_1\frac{\partial U_2}{\partial w_2}, & x_2\frac{\partial U_2}{\partial x_2} + y_2\frac{\partial U_2}{\partial y_2} + z_2\frac{\partial U_2}{\partial z_2} + w_2\frac{\partial U_2}{\partial w_2} \end{vmatrix}$$

$$= \tfrac{1}{4}\Big[(y_1z_2)\frac{\partial(U_1, U_2)}{\partial(y_1, z_2)} + (z_1x_2)\frac{\partial(U_1, U_2)}{\partial(z_1, x_2)} + (x_1y_2)\frac{\partial(U_1, U_2)}{\partial(x_1, y_2)} + (x_1w_2)\frac{\partial(U_1, U_2)}{\partial(x_1, w_2)}$$
$$+ (y_1w_2)\frac{\partial(U_1, U_2)}{\partial(y_1, w_2)} + (z_1w_2)\frac{\partial(U_1, U_2)}{\partial(z_1, w_2)}\Big],$$

where

$$\frac{\partial(U_1, U_2)}{\partial(y_1, z_2)} = \begin{vmatrix} \frac{\partial U_1}{\partial y_1}, & \frac{\partial U_1}{\partial z_1} \\ \frac{\partial U_2}{\partial y_2}, & \frac{\partial U_2}{\partial z_2} \end{vmatrix}, \text{ etc.}$$

i. e.

$$\tfrac{1}{4}\frac{\partial(U_1, U_2)}{\partial(y_1, z_2)} = \begin{vmatrix} hx_1 + by_1 + fz_1 + mw_1, & gx_1 + fy_1 + cz_1 + nw_1 \\ hx_2 + by_2 + fz_2 + mw_2, & gx_2 + fy_2 + cz_2 + nw_2 \end{vmatrix}$$

$$= (bc - f^2)(y_1z_2) + (fg - ch)(z_1x_2) + (fh - bg)(x_1y_2) + (hn - gm)(x_1w_2)$$
$$+ (bn - fm)(y_1w_2) + (fn - cm)(z_1w_2)$$
$$= \rho\,[(bc - f^2)(\alpha\delta') + (fg - ch)(\beta\delta') + (fh - bg)(\gamma\delta') + (hn - gm)(\beta\gamma')$$
$$+ (bn - fm)(\gamma\alpha') + (fn - cm)(\alpha\beta')]$$

$$= \rho\begin{vmatrix} h & g & \alpha & \alpha' \\ b & f & \beta & \beta' \\ f & c & \gamma & \gamma' \\ m & n & \delta & \delta' \end{vmatrix}, \text{ etc.}$$

and substituting the values of (y_1z_2), etc., and $\tfrac{1}{4}\dfrac{\partial(U_1, U_2)}{\partial(y_1, z_2)}$, etc., we find

$$U_{11}U_{22} - U_{12}^2 = \rho^2\Big[(\alpha\delta')\begin{vmatrix} h & g & \alpha & \alpha' \\ b & f & \beta & \beta' \\ f & c & \gamma & \gamma' \\ m & n & \delta & \delta' \end{vmatrix} - (\beta\delta')\begin{vmatrix} a & g & \alpha & \alpha' \\ h & f & \beta & \beta' \\ g & c & \gamma & \gamma' \\ l & n & \delta & \delta' \end{vmatrix}$$

* Salmon's Geometry of Three Dimensions, § 51.

$$+(\gamma\delta')\begin{vmatrix} a & h & \alpha & \alpha' \\ h & b & \beta & \beta' \\ g & f & \gamma & \gamma' \\ l & m & \delta & \delta' \end{vmatrix} + (\beta\gamma')\begin{vmatrix} a & l & \alpha & \alpha' \\ h & m & \beta & \beta' \\ g & n & \gamma & \gamma' \\ l & d & \delta & \delta' \end{vmatrix} + (\gamma\alpha')\begin{vmatrix} h & l & \alpha & \alpha' \\ b & m & \beta & \beta' \\ f & n & \gamma & \gamma' \\ m & d & \delta & \delta' \end{vmatrix}$$

$$+(\alpha\beta')\begin{vmatrix} g & l & \alpha & \alpha' \\ f & m & \beta & \beta' \\ c & n & \gamma & \gamma' \\ n & d & \delta & \delta' \end{vmatrix}\Bigg] = \rho^2\Delta_{PP'}, \text{ where } \Delta_{PP'} = (abcd)_{PP'} = \begin{vmatrix} a & h & g & l & \alpha & \alpha' \\ h & b & f & m & \beta & \beta' \\ g & f & c & n & \gamma & \gamma' \\ l & m & n & d & \delta & \delta' \\ \alpha & \beta & \gamma & \delta & 0 & 0 \\ \alpha' & \beta' & \gamma' & \delta' & 0 & 0 \end{vmatrix},$$

i. e. Δ, the determinant of U, bordered by the coefficients of P and P'; so that $U_{11}U_{22} - U_{12}^2$ and $\Delta_{PP'}$ have always the same sign when not evanescent, and vanish together.

The intersections of U and the line PP' are then real and distinct, real and coincident, or conjugate imaginary according as $\Delta_{PP'} <, =,$ or > 0.

The condition that the line PP' touches U is then $\Delta_{PP'} = 0$.*

VI. The nature of the intersection of a quadric surface U with a plane P, defined as in V., can be determined by eliminating one variable and examining the nature of the conic represented by the resultant. Eliminating w the resultant is

$$\begin{aligned}&(a\delta^2 - 2l\alpha\delta + d\alpha^2)x^2 + (b\delta^2 - 2m\beta\delta + d\beta^2)y^2 + (c\delta^2 - 2n\gamma\delta + d\gamma^2)z^2 \\ &+ 2(f\delta^2 - m\gamma\delta - n\beta\delta + d\beta\gamma)yz + 2(g\delta^2 - l\gamma\delta - n\alpha\delta + d\alpha\gamma)zx \\ &+ 2(h\delta^2 - l\beta\delta - m\alpha\delta + d\alpha\beta)xy = 0,\end{aligned}$$

the determinant of which is

$$\begin{vmatrix} a\delta^2 - 2l\alpha\delta + d\alpha^2, & h\delta^2 - l\beta\delta - m\alpha\delta + d\alpha\beta, & g\delta^2 - l\gamma\delta - n\alpha\delta + d\alpha\gamma \\ h\delta^2 - l\beta\delta - m\alpha\delta + d\alpha\beta, & b\delta^2 - 2m\beta\delta + d\beta^2, & f\delta^2 - m\gamma\delta - n\beta\delta + d\beta\gamma \\ g\delta^2 - l\gamma\delta - n\alpha\delta + d\alpha\gamma, & f\delta^2 - m\gamma\delta - n\beta\delta + d\beta\gamma, & c\delta^2 - 2n\gamma\delta + d\gamma^2 \end{vmatrix}$$

$$= -\delta^4\Delta_P,$$

where

$$\Delta_P = (abcd)_P = \begin{vmatrix} a & h & g & l & \alpha \\ h & b & f & m & \beta \\ g & f & c & n & \gamma \\ l & m & n & d & \delta \\ \alpha & \beta & \gamma & \delta & 0 \end{vmatrix}$$

* Salmon's Geometry of Three Dimensions, § 80.

[namely any determinant with the suffix P is bordered by the coefficients of P]; the principal first minors are $-\delta^2(bcd)_P$, $-\delta^2(acd)_P$, $-\delta^2(abd)_P$, where

$$(bcd)_P = \begin{vmatrix} b & f & m & \beta \\ f & c & n & \gamma \\ m & n & d & \delta \\ \beta & \gamma & \delta & 0 \end{vmatrix}, \quad (acd)_P = \begin{vmatrix} a & g & l & \alpha \\ g & c & n & \gamma \\ l & n & d & \delta \\ \alpha & \gamma & \delta & 0 \end{vmatrix}, \quad (abd)_P = \begin{vmatrix} a & h & l & \alpha \\ h & b & m & \beta \\ l & m & d & \delta \\ \alpha & \beta & \delta & 0 \end{vmatrix},$$

and the principal constituents are $-(ad)_P$, $-(bd)_P$, $-(cd)_P$, where

$$(ad)_P = \begin{vmatrix} a & l & \alpha \\ l & d & \delta \\ \alpha & \delta & 0 \end{vmatrix}, \quad (bd)_P = \begin{vmatrix} b & m & \beta \\ m & d & \delta \\ \beta & \delta & 0 \end{vmatrix}, \quad (cd)_P = \begin{vmatrix} c & n & \gamma \\ n & d & \delta \\ \gamma & \delta & 0 \end{vmatrix}.$$

The results of eliminating x, y, or z, instead of w, from $U = 0$ and $P = 0$ are evident from the above; then by III., if we introduce also

$$(abc)_P = \begin{vmatrix} a & h & g & \alpha \\ h & b & f & \beta \\ g & f & c & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix}, \quad (bc)_P = \begin{vmatrix} b & f & \beta \\ f & c & \gamma \\ \beta & \gamma & 0 \end{vmatrix}, \quad (ac)_P = \begin{vmatrix} a & g & \alpha \\ g & c & \gamma \\ \alpha & \gamma & 0 \end{vmatrix}, \quad (ab)_P = \begin{vmatrix} a & h & \alpha \\ h & b & \beta \\ \alpha & \beta & 0 \end{vmatrix},$$

the intersection of P with U will be:

an imaginary true conic, if $(ab)_P(abcd)_P > 0$ and $(abc)_P < 0$;

a real true conic, if $(ab)_P(abcd)_P \leqq 0$ or $(abc)_P \geqq 0$, but $(abcd)_P \lessgtr 0$;

a pair of conjugate imaginary lines, if $(abcd)_P = 0$ and either $(bcd)_P < 0$ or $(acd)_P < 0$ or $(abd)_P < 0$ or $(abc)_P < 0$;

a pair of distinct real lines, if $(abcd)_P = 0$ and either $(bcd)_P > 0$ or $(acd)_P > 0$ or $(abd)_P > 0$ or $(abc)_P > 0$;

a pair of coincident real lines, if $(abcd)_P = 0$, and $(bcd)_P = (acd)_P = (abd)_P = (abc)_P = 0$.

The condition that P touches U is then $(abcd)_P = 0$.*

If $(ab)_P\Delta_P > 0$ and $(abc)_P < 0$, then also $(ac)_P\Delta_P > 0$, $(ad)_P\Delta_P > 0$, $(bc)_P\Delta_P > 0$, $(bd)_P\Delta_P > 0$, $(cd)_P\Delta_P > 0$, $(bcd)_P < 0$, $(acd)_P < 0$, and $(abd)_P < 0$; and if $\Delta_P = 0$, then those of the determinants $(bcd)_P$, $(acd)_P$, $(abd)_P$, and $(abc)_P$, which do not vanish have the same sign.

VII. The nature of a quadric U is determined by a method analogous to that of III., *i. e.* by considering the nature of the intersections of U with the planes

$$x + \rho w = 0,\; y + \rho w = 0,\; z + \rho w = 0,\; y + \rho z = 0,\; x + \rho z = 0,\; x + \rho y = 0$$

for all real values of ρ. Namely, in order to avoid the consequences of a special

* Salmon's Geometry of Three Dimensions, § 79.

relation of U to the tetrahedron of reference, it is necessary to consider the intersections of at least five of these pencils of planes. The results for all these pencils can be obtained from those for any one by an interchange of letters. Let

$$U \equiv ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw,$$

$$\Delta = (abcd) = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}, \quad A = (bcd) = \begin{vmatrix} b & f & m \\ f & c & n \\ m & n & d \end{vmatrix},$$

$$B = (acd) = \begin{vmatrix} a & g & l \\ g & c & n \\ l & n & d \end{vmatrix}, \quad C = (abd) = \begin{vmatrix} a & h & l \\ h & b & m \\ l & m & d \end{vmatrix}, \quad D = (abc) = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

$$F = -\begin{vmatrix} a & h & l \\ g & f & n \\ l & m & d \end{vmatrix}, \quad L = -\begin{vmatrix} h & b & f \\ g & f & c \\ l & m & n \end{vmatrix}, \text{ etc.,}$$

$$(ad) = \begin{vmatrix} a & l \\ l & d \end{vmatrix}, \ (bd) = \begin{vmatrix} b & m \\ m & d \end{vmatrix}, \ (cd) = \begin{vmatrix} c & n \\ n & d \end{vmatrix}, \ (bc) = \begin{vmatrix} b & f \\ f & c \end{vmatrix}, \ (ac) = \begin{vmatrix} a & g \\ g & c \end{vmatrix}, \ (ab) = \begin{vmatrix} a & h \\ h & b \end{vmatrix}.$$

The nature of the intersection of U with the plane $x + \rho w$ is determined by the criteria of VI. We have here

$$P \equiv x + \rho w, \ (abcd)_P = -(A + 2\rho L + \rho^2 D), \ (bcd)_P = -\rho^2 (bc), \ (abc)_P = -(bc),$$
$$-(acd)_P = (cd) + 2\rho(gn - cl) + \rho^2(ac),$$
$$-(abd)_P = (bd) + 2\rho(hm - bl) + \rho^2(ab), \ (ab)_P = -b.$$

1. The intersection with every real plane $x + \rho w$ is then an imaginary true conic if $b(A + 2\rho L + \rho^2 D) > 0$ for every real value of ρ and $(bc) > 0$. Then $AD - L^2 = (bc)\Delta > 0$, so that neither A nor D vanishes, and $bD > 0$. The conditions are therefore

$$(abcd) > 0, \ b(abc) > 0, \ (bc) > 0.$$

2. The intersection is an imaginary true conic for every real plane $x + \rho w$ but one, and a pair of imaginary lines for this one, if $b(A + 2\rho L + \rho^2 D) \overline{\gtrless} 0$ for every real value of ρ and $(bc) > 0$. Then $AD - L^2 = (bc)\Delta = 0$, but neither A nor D vanishes, and $bD > 0$. The conditions are therefore

$$(abcd) = 0, \ b(abc) > 0, \ (bc) > 0.$$

3. The intersection is a real true conic for every real plane $x + \rho w$ if $b(A + 2\rho L + \rho^2 D) \overline{\gtrless} 0$ for every real value of ρ or $(bc) \overline{\gtrless} 0$, but not $A + 2\rho L + \rho^2 D = 0$ for any real value of ρ. Then $AD - L^2 = (bc)\Delta > 0$, so that neither A nor D vanishes; also, if $b(A + 2\rho L + \rho^2 D) \overline{\gtrless} 0$, then $bD \overline{\gtrless} 0$. There are two

possible sets of conditions:

either $(abcd) > 0, \quad b(abc) \leqq 0, \quad (bc) > 0,$

or $(abcd) < 0, \quad bc < 0.$

4. The intersection is a real true conic for every real plane $x + \rho w$ but one, say $P_1 \equiv x + \rho_1 w$, and a pair of distinct real lines for this one, if $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$, and either $b(A + 2\rho L + \rho^2 D) \leqq 0$ for every real value of ρ and $(bc) = 0$ and either $-(abd)_{P_1} < 0$ or $-(acd)_{P_1} < 0$, or $(bc) < 0$. Then $AD - L^2 = (bc)\Delta = 0$. If $(bc) = 0$ it is found that $\rho_1 = -\dfrac{A}{L} = -\dfrac{L}{D} = \dfrac{cm - fn}{ch - fg}$, whence*

$$-(ch - fg)^2 (acd)_{P_1} = (ch - fg)^2 [(cd) + 2\rho_1 (gn - cl) + \rho_1^2 (ac)] = -c^2\Delta,$$
$$-(ch - fg)^2 (abd)_{P_1} = (ch - fg)^2 [(bd) + 2\rho_1 (hm - bl) + \rho_1^2 (ab)] = -b^2\Delta,$$

thus $\Delta > 0$;† also, since $b(A + 2\rho L + \rho^2 D) \leqq 0$, $bA \leqq 0$ and $bD \leqq 0$, but not at once, $A = D = 0$, since $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$. There are two possible sets of conditions: either

$(abcd) > 0, \quad b(abc) \leqq 0, \quad b(bcd) \leqq 0,$ but not at once $(abc) = (bcd) = 0, \quad (bc) = 0;$

or $(abcd) = 0, \quad (bc) < 0,$ but not at once $(abc) = (bcd) = 0.$

5. The intersection is a real true conic for every real plane $x + \rho w$ but one, say $P_1 \equiv x + \rho_1 w$, and a pair of imaginary lines for this one, if $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$, and either $b(A + 2\rho L + \rho^2 D) \leqq 0$ for every real value of ρ and $(bc) = 0$ and either $-(abd)_{P_1} > 0$ or $-(acd)_{P_1} > 0$, or $(bc) > 0$. Then $AD - L^2 = (bc)\Delta = 0$. If $(bc) = 0$, we have, as in 4, $\rho_1 = \dfrac{cm - fn}{ch - fg}$, $-(ch - fg)^2 (acd)_{P_1} = -c^2\Delta$, $-(ch - fg)^2 (abd)_{P_1} = -b^2\Delta$, $\therefore \Delta < 0$;‡ also, as in 4, $bA \leqq 0$ and $bD \leqq 0$, but not at once $A = D = 0$. There are two sets of conditions: either

$(abcd) < 0, \; b(abc) \leqq 0, \; b(bcd) \leqq 0,$ but not at once $(abc) = (bcd) = 0, \; (bc) = 0,$

or $(abcd) = 0, \; b(abc) \leqq 0, \; b(bcd) \leqq 0,$ but not at once $(abc) = (bcd) = 0, \; (bc) > 0.$

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* The cases in which $(bc) = 0$ and either $c = 0$, or $b = 0$, or $c \lessgtr 0$ and $ch - fg = 0$ require special consideration, but they lead to the same conditions as the general case.

If $c = 0$, then $f = 0$, $\rho_1 = \frac{n}{g}$, $(acd)_{P_1} = 0$, $-(abd)_{P_1} = -\frac{\Delta}{g^2}$, $\therefore \Delta > 0$, $A = -bn^2$, $D = -bg^2$, $bA = -b^2 n^2$, $bD = -b^2 g^2$.

If $b = 0$, we have conditions analogous to those for $c = 0$.

If $c \lessgtr 0$ and $ch - fg = 0$, then $b = \frac{f^2}{c}$, $h = \frac{fg}{c}$, $\rho_1 = \infty$, $(ab) = \frac{f^2}{c^2}(ac)$, $-(abd)_{P_1}$ and $-(acd)_{P_1}$ are of the same sign as (ac), $\therefore (ac) < 0$, $\Delta = -\frac{(ac)}{c^2}(fn - cm)^2 > 0$, $A = -\frac{1}{c}(fn - cm)^2$, $D = 0$, $ba = -\frac{f^2}{c^2}(fn - cm)^2 \leqq 0$.

† It cannot be that $b = c = 0$, for then $A = D = 0$, which is excluded by the next line.

‡ See also the first footnote to 4.

6. The intersection is a real true conic for every real plane $x + \rho w$ but one, say $P_1 \equiv x + \rho_1 w$, and a pair of coincident real lines for this one, if $A + 2\rho L + \rho^2 D = 0$ only for $\rho = \rho_1$, and $b(A + 2\rho L + \rho^2 D) \overline{\gtrless} 0$ for every real value of ρ, $(bc) = 0$, $(abd)_{P_1} = (acd)_{P_1} = 0$. Then $AD - L^2 = (bc)\,\Delta = 0$, $\rho_1 = \dfrac{cm - fn}{ch - fg}$, $-(ch - fg)^2 (acd)_{P_1} = -c^2\Delta$, $-(ch - fg)^2 (abd)_{P_1} = -b^2\Delta$, $\therefore \Delta = 0$;* also $bA \overline{\gtrless} 0$ and $bD \overline{\gtrless} 0$, but not at once $A = D = 0$. The conditions are

$$(abcd) = 0,\ b(abc) \overline{\gtrless} 0,\ b(bcd) \overline{\gtrless} 0, \text{ but not at once } (abc) = (bcd) = 0,\ (bc) = 0.$$

7. The intersection is a real true conic for every real plane $x + \rho w$ but two if $A + 2\rho L + \rho^2 D = 0$ for two real values of ρ, and either $b(A + 2\rho L + \rho^2 D) \overline{\gtrless} 0$ for every real value of ρ or $(bc) < 0$. Then $AD - L^2 = (bc)\,\Delta < 0$, and hence $(bc) \lesseqgtr 0$. Since $A + 2\rho L + \rho^2 D$ changes sign for real values of ρ, $b(A + 2\rho L + \rho^2 D) \overline{\gtrless} 0$ only if $b = 0$, but then $(bc) < 0$. The conditions are therefore

$$(abcd) > 0,\ (bc) < 0.$$

8. The intersection is an imaginary true conic for an infinite succession of real planes $x + \rho w$, a real true conic for an infinite succession of real planes $x + \rho w$, and a pair of imaginary straight lines for each of the two real planes which separate these successions, if $AD - L^2 = (bc)\Delta < 0$ and $(bc) > 0$, *i. e.* if

$$(abcd) < 0,\ (bc) > 0.$$

9. The intersection is a pair of imaginary lines for every real plane $x + \rho w$ if $A = L = D = 0$ and either $(bc) > 0$, or $(bc) = 0$, $-(acd)_P \overline{\gtrless} 0$ and $-(abd)_P \overline{\gtrless} 0$ for every real value of ρ, but not at once $(abd)_P = (acd)_P = 0$. Hence, if $(bc) = 0$, $(ac)(cd) - (gn - cl)^2 = cB \overline{\gtrless} 0$ and $(ab)(bd) - (hm - bl)^2 = bC \overline{\gtrless} 0$, $(ab) > 0$ or $(ac) > 0$ or $(bd) > 0$ or $(cd) > 0$, together with such other conditions as shall ensure the inequality of the values of ρ for which $(abd)_P$ and $(acd)_P$ vanish, if the latter are both perfect squares, *i. e.* neither of them can be a finite (or vanishing) multiple of the other if $bC = cB = 0$. Now if $(bc) = 0$ and $A = D = 0$, we have, if $c \lesseqgtr 0$, $b = \dfrac{f^2}{c}$, $A = -\dfrac{1}{c}(cm - fn)^2$, $D = -\dfrac{1}{c}(ch - fg)^2$, $\therefore\ m = \dfrac{fn}{c}$, $h = \dfrac{fg}{c}$, $(ab) = \dfrac{f^2}{c^2}(ac)$, $(bd) = \dfrac{f^2}{c^2}(cd)$, $hm - bl^2 = \dfrac{f^2}{c^2}(gn - cl)$, $(abd)_P = \dfrac{f^2}{c^2}(acd)_P$; if $c = 0$, we have $f = 0$, $A = -bn^2$, $D = -bg^2$, $\therefore g = n = 0$ (since $b = c = 0$ makes $(ab) \overline{\gtrless} 0$, $(ac) \overline{\gtrless} 0$, $(bd) \overline{\gtrless} 0$, $(cd) \overline{\gtrless} 0$), $(ac) = (cd) = 0$, $gn - cl = 0$, $(acd)_P \equiv 0$; in each of these cases one of the quantities $(abd)_P$ and $(acd)_P$ is a finite (or vanishing) multiple of the other, therefore we cannot have at

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* See also the first footnote to 4.

once $bC = cB = 0$, but must have either $bC > 0$ or $cB > 0$. If either of the above sets of conditions are satisfied it will be found that $\Delta = 0$; also, if $A = D = 0$ and either $\Delta = 0$ or $(bc) = 0$, then $L = 0$. The conditions for this species of intersection are then

either $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ (bc) > 0;$

or $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) > 0$ or $c(acd) > 0,\ (bc) = 0,$
$(ab) > 0$ or $(ac) > 0$ or $(bd) > 0$ or $(cd) > 0$.

10. The intersection is a pair of imaginary lines for every real plane $x + \rho w$ but one, and a pair of coincident real lines for that one, if $A = D = L = 0$, $(bc) = 0$, $-(acd)_P \overline{\gtrless} 0$ and $-(abd)_P \overline{\gtrless} 0$ for every real value of ρ, namely $(abd)_P = (acd)_P = 0$ for one and only one real value of ρ. Then, as is evident from the treatment of the preceding case, the conditions for this species of intersection are

$(abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) = c(acd) = 0,\ (bc) = 0,\ (ab) > 0$ or $(ac) > 0$ or $(bd) > 0$ or $(cd) > 0$.

11. The intersection is a pair of distinct real lines for every real plane $x + \rho w$, if $A = D = L = 0$ and either $(bc) < 0$, or $(bc) = 0$ and $-(abd)_P \overline{\lessgtr} 0$ and $-(acd)_P \overline{\lessgtr} 0$ for every real value of ρ, but not at once $(abd)_P = (acd)_P = 0$. If $(bc) = 0$, the conditions are the same as in 9, excepting that here $(ab) < 0$ or $(ac) < 0$ or $(bd) < 0$ or $(cd) < 0$, and hence $b = c = f = 0$ is possible; if then $b = c = f = 0$, $-(acd)_P \equiv -(n - \rho g)^2$, $-(abd)_P \equiv -(m - \rho h)^2$, and we must have $gm - hn \lessgtr 0$, but we also have $\Delta = (gm - hn)^2$, hence $\Delta > 0$. The conditions for this species of intersection are then

either $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ (bc) < 0;$

or $\quad (abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) > 0$ or $c(acd) > 0,\ (bc) = 0,$
$(ab) < 0$ or $(ac) < 0$ or $(bd) < 0$ or $(cd) < 0;$

or $\quad (abcd) > 0,\ (abc) = (bcd) = 0,\ (bc) = 0.$

12. The intersection is a pair of distinct real lines for every real plane $x + \rho w$ but one, and a pair of coincident real lines for that one, if $A = D = L = 0$, $(bc) = 0$, $-(abd)_P \overline{\lessgtr} 0$ and $-(acd)_P \overline{\lessgtr} 0$ for every real value of ρ, namely $(abd)_P = (acd)_P = 0$ for one and only one real value of ρ. The conditions are then

$(abcd) = 0,\ (abc) = (bcd) = 0,\ b(abd) = c(acd) = 0,\ (bc) = 0,\ (ab) < 0$ or $(ac) < 0$ or $(bd) < 0$ or $(cd) < 0$.

13. The intersection is a pair of imaginary lines for an infinite succession of real planes $x + \rho w$, a pair of distinct real lines for an infinite succession of real planes $x + \rho w$, and a pair of coincident real lines for each of the two real

planes which separate these successions, if $A=D=L=0$, $(bc)=0$, $bC<0$ or $cB<0$. Then, as in 9, since b and c cannot both vanish, $\Delta=0$. Hence the conditions are

$$(abcd)=0,\ (abc)=(bcd)=0,\ b(abd)<0 \text{ or } c(acd)<0,\ (bc)=0.$$

14. The intersection is a pair of coincident real lines for every real plane $x+\rho w$, if $A=D=L=0$, $(bc)=0$, $b(abd)=c(acd)=0$, $(ab)=(ac)=(bd)=(cd)=0$. Then $\Delta=0$, and the conditions are

$$(abcd)=0,\ (abc)=(bcd)=0,\ b(abd)=c(acd)=0,\ (bc)=0,\ (ab)=(ac)=(bd)=(cd)=0.$$

These results tabulated are:

<table>
<tr><th>$(abcd)$</th><th>(bc)</th><th>$b\,(abc)$
$b\,(bcd)$</th><th>(abc)
(bcd)</th><th>$b\,(abd)$
$c\,(acd)$</th><th>(ab), (ac)
(bd), (cd)</th><th>Species</th></tr>
<tr><td rowspan="5">>0</td><td rowspan="2">>0</td><td>either >0</td><td></td><td></td><td></td><td>1</td></tr>
<tr><td>either $\leqq 0$</td><td></td><td></td><td></td><td>3</td></tr>
<tr><td><0</td><td></td><td></td><td></td><td></td><td>7</td></tr>
<tr><td rowspan="2">$=0$</td><td rowspan="2"></td><td>either $\lessgtr 0$</td><td></td><td></td><td>4</td></tr>
<tr><td>both $=0$</td><td></td><td></td><td>11</td></tr>
<tr><td rowspan="3"><0</td><td>>0</td><td></td><td></td><td></td><td></td><td>8</td></tr>
<tr><td><0</td><td></td><td></td><td></td><td></td><td>3</td></tr>
<tr><td>$=0$</td><td></td><td></td><td></td><td></td><td>5</td></tr>
<tr><td rowspan="12">$=0$</td><td rowspan="3">>0</td><td>either >0</td><td></td><td></td><td></td><td>2</td></tr>
<tr><td>either $\leqq 0$</td><td>either $\lessgtr 0$</td><td></td><td></td><td>5</td></tr>
<tr><td></td><td>both $=0$</td><td></td><td></td><td>9</td></tr>
<tr><td rowspan="2"><0</td><td rowspan="2"></td><td>either $\lessgtr 0$</td><td></td><td></td><td>4</td></tr>
<tr><td>both $=0$</td><td></td><td></td><td>11</td></tr>
<tr><td rowspan="7">$=0$</td><td rowspan="7"></td><td>either $\lessgtr 0$</td><td></td><td></td><td>6</td></tr>
<tr><td rowspan="6">both $=0$</td><td rowspan="2">either >0</td><td>either >0</td><td>9</td></tr>
<tr><td>either <0</td><td>11</td></tr>
<tr><td>either <0</td><td></td><td>13</td></tr>
<tr><td rowspan="3">both $=0$</td><td>either >0</td><td>10</td></tr>
<tr><td>either <0</td><td>12</td></tr>
<tr><td>all $=0$</td><td>14</td></tr>
</table>

The interchanges of letters by which this table is made to give the conditions for the different species of intersections with planes through either edge of the tetrahedron of reference are evident.

There are the following eight absolute classes of quadric surfaces (surfaces of the second order given by real equations in point-coordinates):

imaginary true quadrics,
convex quadrics (real true quadrics with imaginary generators),
real quadric scrolls (with real generators),
imaginary quadric cones (with real vertices),
real quadric cones,
pairs of imaginary planes (with real double edges),
pairs of distinct real planes,
pairs of coincident real planes.

For an imaginary true quadric the intersections are of the species 1 for each edge of the tetrahedron, and the necessary and sufficient conditions for this class of quadrics may be written

$$(abcd)>0,\ (ab)>0,\ a(abc)>0.$$

A convex quadric may be cut by either edge of the tetrahedron in distinct real points, in coincident real points (case of tangency), or in imaginary points; the intersections by a plane rotating about the edge are of the species 3, 5 or 8, which concur only if $(abcd)<0$, which is therefore the condition for this class.

A real quadric scroll may be cut by either edge in distinct real points, in coincident real points (case of tangency), or in imaginary points, or it may contain the edge as generator; the intersections by the rotating plane may be of the species 7, 4, 3 or 11, which concur only if $(abcd)>0$, but the other conditions for 1 must not all be satisfied, *i. e.* the conditions for this class are

$$(abcd)>0, \text{ and either } (ab)\overline{\lessdot}\,0 \text{ or } a(abc)\overline{\lessdot}\,0.$$

An imaginary quadric cone may be cut by either edge of the tetrahedron in imaginary points or in coincident real points (the edge passing through the vertex of the cone), but not more than three edges can have the latter position; the intersections by the revolving plane are of the species 2 or 9; and, conversely, if the intersections are of the species 2 for either edge, the quadric is evidently an imaginary cone. The conditions are therefore

$$(abcd)=0, \text{ and either } a(abc)>0 \text{ and } (ab)>0, \text{ or } a(abd)>0 \text{ and } (ab)>0,$$
$$\text{or } c(acd)>0 \text{ and } (cd)>0, \text{ or } c(bcd)>0 \text{ and } (cd)>0.$$

A real cone is met by any edge of the tetrahedron in imaginary points, in distinct real points, or in coincident real points so that the edge is tangent to the cone or passes through the vertex, or the cone contains the edge as a generator; the intersections with the revolving plane are of the species 5, 4, 6, 13, 11 or 12, namely, of the species 12 for at most three edges meeting in one point. Now by a real linear transformation which does not alter the value of Δ, any tangent to the cone can be made an edge of the tetrahedron of reference, hence $\Delta = 0$. Then the conditions are evidently

$$(abcd) = 0, \text{ either } (ab) \leqq 0, \text{ or } a(abc) \leqq 0 \text{ and } a(abd) \leqq 0,$$

and either $(cd) \leqq 0$, or $c(acd) \leqq 0$, and $c(bcd) \leqq 0$.

A pair of imaginary planes is met by any edge of the tetrahedron in two imaginary points, or in two coincident real points, or the edge lies entirely in the quadric (is the double line of the pair of planes), but only one edge can have the latter position; the intersections with the revolving plane are of the species 9, 10 or 14, namely, of the species 9 for at least one edge, and of the species 14 for at most one edge. The conditions are therefore

$$(abcd) = 0, \ (bcd) = (acd) = (abd) = (abc) = 0, \text{ and either } (ab) > 0$$
$$\text{or } (ac) > 0 \text{ or } (ad) > 0 \text{ or } (bc) > 0 \text{ or } (bd) > 0 \text{ or } (cd) > 0.$$

A pair of distinct real planes is met by any edge of the tetrahedron in two distinct real points, or in two coincident real points, or the edge lies in one or both of the planes of the pair, and at least one edge must have the first position and at most one edge can have the last position (coincident with the double line of the pair of planes); the intersections with the revolving plane are of the species 11, 12 or 14, namely, of the species 11 for at least one edge, and of the species 14 for at most one edge. By a real linear transformation which does not alter the value of Δ, the double line of the pair of planes can be made an edge of the tetrahedron, hence $\Delta = 0$. The conditions are then

$$(abcd) = 0, \ (bcd) = (acd) = (abd) = (abc) = 0, \text{ and either } (ab) < 0$$
$$\text{or } (ac) < 0 \text{ or } (ad) < 0 \text{ or } (bc) < 0 \text{ or } (bd) < 0 \text{ or } (cd) < 0.$$

A pair of coincident real planes is met by any edge of the tetrahedron in two coincident real points; the intersections with the revolving plane are of the species 14 for every edge. The conditions are therefore

$$(abcd) = 0, \ (bcd) = (acd) = (abd) = (abc) = 0,$$
$$(ab) = (ac) = (ad) = (bc) = (bd) = (cd) = 0.$$

The conditions here given for the different classes are mutually exclusive and therefore not only necessary but sufficient. They can be put into a somewhat

simpler form. Any succession of principal minors of Δ (including Δ itself) of which each is a principal first minor of the preceding may be called a *sequence* of principal minors of Δ, and the minors themselves may be denoted by (4), (3), (2), (1) respectively, the number enclosed in parentheses denoting the order of the determinant constituting the minor; thus there is a sequence in which $(4) = (abcd)$, $(3) = (abd)$, $(2) = (bd)$, $(1) = b$; (4) is always $(abcd)$, and every (1) is a principal constituent of $(abcd)$. The conditions for the different classes are then the following:

if $(4) > 0$ and there is a sequence in which $(1)(3) > 0$ and $(2) > 0$, U is an imaginary true quadric;

if $(4) > 0$ and there is no sequence in which $(1)(3) > 0$ and $(2) > 0$, U is a quadric scroll;

if $(4) < 0$, U is a convex quadric;

if $(4) = 0$ and there is a sequence in which $(1)(3) > 0$ and $(2) > 0$, U is an imaginary cone;

if $(4) = 0$ and there is a sequence in which either $(1)(3) \overset{=}{<} 0$ or $(2) \overset{=}{<} 0$, but not $(3) = 0$, U is a real cone;

if $(4) = 0$ and in every sequence $(3) = 0$, and in at least one sequence $(2) > 0$, U is a pair of imaginary planes;

if $(4) = 0$ and in every sequence $(3) = 0$, and in at least one sequence $(2) < 0$, U is a pair of distinct real planes;

if $(4) = 0$ and in every sequence $(3) = 0$ and $(2) = 0$, U is a pair of coincident real planes.

(TO BE CONTINUED.)

The Imaginary Period in Elliptic Functions.

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1. The object of this paper is to present a proof of the imaginary period of the elliptic functions, which appears more in accordance, than that usually given, with the manner in which the notion of a function of an imaginary variable naturally presents itself after the study of the function of the real argument. In doing this I shall employ, and to some extent develope, a notation which I have found useful in simplifying the processes and abridging the expression of results in the elementary treatment of the elliptic functions.

2. If we put
$$\operatorname{sn} u = \frac{s}{n}, \quad \operatorname{cn} u = \frac{c}{n}, \quad \operatorname{dn} u = \frac{d}{n},$$
we may consider the ratios of s, c, d and n which are functions of u, without defining the actual values of these quantities as functions of u;* all equations involving these quantities being thus homogeneous. For example, the relations between the squares of $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$ are expressed by
$$s^2 + c^2 = n^2, \tag{1}$$
and
$$k^2 s^2 + d^2 = n^2; \tag{2}$$
whence, by elimination, we have also
$$k^2 c^2 + k'^2 n^2 = d^2, \tag{3}$$
and
$$c^2 + k'^2 s^2 = d^2, \tag{4}$$
of which (3) expresses the relation between the squares of $\operatorname{cn} u$ and $\operatorname{dn} u$.

3. In accordance with this notation the first letter of the functional sign sn, cn, or dn points out the numerator, and the second the denominator of the ratio, and accordingly we are also led to put
$$\operatorname{sc} u = \frac{s}{c} = \frac{\operatorname{sn} u}{\operatorname{cn} u}, \quad \operatorname{ns} u = \frac{n}{s} = \frac{1}{\operatorname{sn} u}, \text{ etc.},$$

* The quantities s, c, d and n of course may be taken as differing from the theta functions only by certain multipliers involving k, in fact they may be taken to be the actual values of the *tabular* theta functions Θ_1, Θ_2, Θ_3, and Θ, whose logarithms will be given in the forthcoming tables of the theta functions.

a notation already employed by Mr. Glaisher, who has pointed out* the advantage of considering the *twelve* elliptic functions, sn, cn, dn, their six ratios and three reciprocals, instead of supplementing sn, cn and dn with the functions of the coamplitude. In the present notation, these twelve functions are the ratios of the several pairs of quantities which can be formed from the four quantities s, c, d and n, and equations (1)–(4) express the linear relations which exist between the squares of any two functions which have a common denominator, thus the relation between $\mathrm{sc}\, u$ and $\mathrm{dc}\, u$ is given by (4), which may be written in the form $1 + k'^2 \mathrm{sc}^2 u = \mathrm{dc}^2 u$.

4. The twelve functions of a special value of u are of course all given at once, when the ratios of s, c, d and n are given; for example, we have

$$\begin{array}{ll} \text{for } u = 0, & s:c:d:n = 0:1:1:1, \\ \text{for } u = K, & s:c:d:n = 1:0:k':1, \\ \text{for } u = 2K, & s:c:d:n = 0:-1:1:1, \\ \text{and for } u = 3K, & s:c:d:n = -1:0:k':1;\dagger \end{array}$$

and when this is done nothing prevents our taking the values of the proportional numbers as the actual values of the letters; the values given above for instance satisfy equations (1)–(4).

5. The quantities s, c, d, n, when considered as functions of u, must be regarded as containing a common undetermined multiplier; their logarithms therefore contain a common undetermined part, namely, the logarithm of the undetermined multiplier, which must be regarded as an arbitrary function of u, and their logarithmic derivatives will likewise contain a common undetermined part. Hence, although neither the derivatives nor the logarithmic derivatives of the quantities s, c, d and n have determinate values, yet the *differences* of the logarithmic derivatives have determinate values. Thus, the definition of the function sn gives

$$\frac{d}{du}\,\mathrm{sn}\, u = \mathrm{cn}\, u\ \mathrm{dn}\, u,$$

or

$$\frac{d}{du}\frac{s}{n} = \frac{c.d}{n^2},$$

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*See *Messenger of Mathematics*, Vol. XI (Oct. 1881), p. 81 and p. 120.

† In like manner we have

$$\begin{array}{ll} \text{for } u = iK', & s:c:d:n = i:1:k:0, \\ \text{and for } u = K + iK', & s:c:d:n = 1:-ik':0:k; \end{array}$$

the four quantities vanishing for the respective values of u, 0, K, $K + iK'$, iK'; so that for each of these values three of the twelve functions become infinite. It is hardly necessary to remark that the want of analogy between the relations of iK' to the elliptic functions and those of K and of $K + iK'$ (see Cayley's *Elliptic Functions*, p. 13, "only it must be borne in mind that K, $K + iK'$ have, K, iK' have *not*, analogous relations to the elliptic functions") disappears when the twelve functions are considered.

which, putting D for $\frac{d}{du}$, may be written in the form

$$\frac{Ds}{s} - \frac{Dn}{n} = \frac{c.d}{s.n}. \tag{5}$$

To obtain the difference between the logarithmic derivatives of c and n, we have, by differentiating equation (1),

$$sDs + cDc = nDn,$$

and eliminating Ds hereby from equation (5), we find

$$\frac{nDn - cDc}{s^2} - \frac{Dn}{n} = \frac{c.d}{s.n},$$

or

$$\frac{n^2 - s^2}{s^2} \cdot \frac{Dn}{n} - \frac{cDc}{s^2} = \frac{c.d}{s.n};$$

whence, since $n^2 - s^2 = c^2$,

$$\frac{Dc}{c} - \frac{Dn}{n} = -\frac{s.d}{c.n}. \tag{6}$$

In like manner, eliminating Ds from equation (5) by means of

$$k^2 sDs + dDd = nDn,$$

the derivative of equation (2), we find

$$\frac{Dd}{d} - \frac{Dn}{n} = -k^2 \frac{s.c}{d.n}. \tag{7}$$

The other differences are now readily obtained from these by subtraction, they are

$$\frac{Ds}{s} - \frac{Dc}{c} = \frac{d.n}{s.c}, \tag{8}$$

$$\frac{Ds}{s} - \frac{Dd}{d} = \frac{c.n}{s.d}, \tag{9}$$

$$\frac{Dc}{c} - \frac{Dd}{d} = -k'^2 \frac{s.n}{c.d}. \tag{10}$$

6. Equations (5)–(10) give in fact the logarithmic derivatives of the twelve functions, and from them we at once have the derivatives of the functions themselves as follows:

$$\frac{d}{du} \operatorname{sn} u = \frac{c.d}{n^2}, \tag{11}$$

$$\frac{d}{du} \operatorname{cn} u = -\frac{s.d}{n^2}, \tag{12}$$

$$\frac{d}{du} \operatorname{dn} u = -k^2 \frac{s.c}{n^2}, \tag{13}$$

$$\frac{d}{du} \operatorname{sd} u = \frac{c.n}{d^2}, \tag{14}$$

$$\frac{d}{du} \operatorname{cd} u = -k'^2 \frac{s.n}{d^2}, \tag{15}$$

$$\frac{d}{du} \operatorname{nd} u = k^2 \frac{s.c}{d^2}, \tag{16}$$

$$\frac{d}{du}\,\mathrm{sc}\,u = \frac{d.n}{c^2}, \tag{17}$$

$$\frac{d}{du}\,\mathrm{dc}\,u = k'^2\frac{s.n}{c^2}, \tag{18}$$

$$\frac{d}{du}\,\mathrm{nc}\,u = \frac{s.d}{c^2}, \tag{19}$$

$$\frac{d}{du}\,\mathrm{cs}\,u = -\frac{d.n}{s^2}, \tag{20}$$

$$\frac{d}{du}\,\mathrm{ds}\,u = -\frac{c.n}{s^2}, \tag{21}$$

$$\frac{d}{du}\,\mathrm{ns}\,u = -\frac{c.d}{s^2}. \tag{22}$$

7. From these equations we readily derive the derivatives of the inverse functions, arg cn x, arg sc x, etc.

Thus, put $$u = \arg\mathrm{cn}\,x,$$

then $$x = \mathrm{cn}\,u = \frac{c}{n},$$

and from (12) $$\frac{dx}{du} = -\frac{s.d}{n^2};$$

expressing s and d in terms of c and n, by means of equations (1) and (3),

$$\frac{dx}{du} = -\frac{\sqrt{(n^2 - c^2 . k^2c^2 + k'^2n^2)}}{n^2} = -\sqrt{(1 - x^2 . k^2x^2 + k'^2)},$$

or $$\frac{d}{dx}\arg\mathrm{cn}\,x = -\frac{1}{\sqrt{(1 - x^2 . k^2x^2 + k'^2)}}.$$

Again, if $$u = \arg\mathrm{sc}\,x,$$

$$x = \mathrm{sc}\,u = \frac{s}{c}.$$

and from (17) $$\frac{dx}{du} = \frac{d.n}{c^2}.$$

Expressing d and n in terms of s and c, by means of equations (4) and (1),

$$\frac{dx}{du} = \frac{\sqrt{(k'^2s^2 + c^2 . s^2 + c^2)}}{c^2} = \sqrt{(k'^2x^2 + 1 . x^2 + 1)},$$

or $$\frac{d}{dx}\arg\mathrm{sc}\,x = \frac{1}{\sqrt{(k'^2x^2 + 1 . x^2 + 1)}}.$$

The complete set of derivatives is as follows:

$$\frac{d}{dx}\arg\mathrm{sn}\,x = \frac{1}{\sqrt{(1 - x^2 . 1 - k^2x^2)}}, \tag{23}$$

$$\frac{d}{dx}\arg\mathrm{cn}\,x = -\frac{1}{\sqrt{(1 - x^2 . k^2x^2 + k'^2)}}, \tag{24}$$

$$\frac{d}{dx}\arg\mathrm{dn}\,x = -\frac{1}{\sqrt{(1 - x^2 . x^2 - k'^2)}}, \tag{25}$$

$$\frac{d}{dx}\arg \mathrm{sd}\, x = \frac{1}{\sqrt{(1-k'^2x^2.k^2x^2+1)}}, \tag{26}$$

$$\frac{d}{dx}\arg \mathrm{cd}\, x = -\frac{1}{\sqrt{(1-x^2.1-k^2x^2)}}, \tag{27}$$

$$\frac{d}{dx}\arg \mathrm{nd}\, x = \frac{1}{\sqrt{(1-k'^2x^2.x^2-1)}}, \tag{28}$$

$$\frac{d}{dx}\arg \mathrm{sc}\, x = \frac{1}{\sqrt{(x^2+1.k'^2x^2+1)}}; \tag{29}$$

$$\frac{d}{dx}\arg \mathrm{dc}\, x = \frac{1}{\sqrt{(x^2-1.x^2-k^2)}}, \tag{30}$$

$$\frac{d}{dx}\arg \mathrm{nc}\, x = \frac{1}{\sqrt{(x^2-1.k'^2x^2+k^2)}}, \tag{31}$$

$$\frac{d}{dx}\arg \mathrm{cs}\, x = -\frac{1}{\sqrt{(x^2+1.x^2+k'^2)}}, \tag{32}$$

$$\frac{d}{dx}\arg \mathrm{ds}\, x = -\frac{1}{\sqrt{(x^2+k^2.x^2-k'^2)}}, \tag{33}$$

$$\frac{d}{dx}\arg \mathrm{ns}\, x = -\frac{1}{\sqrt{(x^2-1.x^2-k^2)}}, \tag{34}$$

in each of which the radical is positive for values of the inverse function between 0 and K.

8. These formulæ give the following expressions for the twelve inverse functions as integrals:

$$\arg \mathrm{sn}\, x = \int_0 \frac{dx}{\sqrt{(1-x^2.1-k^2x^2)}}, \tag{35}$$

$$\arg \mathrm{cn}\, x = -\int_1 \frac{dx}{\sqrt{(1-x^2.k^2x^2+k'^2)}}, \tag{36}$$

$$\arg \mathrm{dn}\, x = -\int_1 \frac{dx}{\sqrt{(1-x^2.x^2-k'^2)}}, \tag{37}$$

$$\arg \mathrm{sd}\, x = \int_0 \frac{dx}{\sqrt{(1-k'^2x^2.k^2x^2+1)}}, \tag{38}$$

$$\arg \mathrm{cd}\, x = -\int_1 \frac{dx}{\sqrt{(1-x^2.1-k^2x^2)}}, \tag{39}$$

$$\arg \mathrm{nd}\, x = \int_1 \frac{dx}{\sqrt{(1-k'^2x^2.x^2-1)}}, \tag{40}$$

$$\arg \mathrm{sc}\, x = \int_0 \frac{dx}{\sqrt{(x^2+1.k'^2x^2+1)}}, \tag{41}$$

$$\arg \mathrm{dc}\, x = \int_1 \frac{dx}{\sqrt{(x^2-1.x^2-k^2)}}, \tag{42}$$

$$\arg \mathrm{nc}\, x = \int_1 \frac{dx}{\sqrt{(x^2-1.k'^2x^2+k^2)}}, \tag{43}$$

$$\arg\ \mathrm{cs}\, x = -\int_\infty \frac{dx}{\sqrt{(x^2+1 . x^2+k'^2)}}, \tag{44}$$

$$\arg\ \mathrm{ds}\, x = -\int_\infty \frac{dx}{\sqrt{(x^2+k^2 . x^2-k'^2)}}, \tag{45}$$

$$\arg\ \mathrm{ns}\, x = -\int_\infty \frac{dx}{\sqrt{(x^2-1 . x^2-k^2)}}. \tag{46}$$

9. It will be noticed that the expressions under the integral sign form six pairs the members of which are either identical or become so when x in one of them is replaced by $k'x$, thus we have

$$\arg\ \mathrm{sn}\, x + \arg\ \mathrm{cd}\, x \ = K, \tag{47}$$

$$\arg\ \mathrm{cn}\, x + \arg\ \mathrm{sd}\, \frac{x}{k'} = K, \tag{48}$$

$$\arg\ \mathrm{dn}\, x + \arg\ \mathrm{nd}\, \frac{x}{k'} = K, \tag{49}$$

$$\arg\ \mathrm{sc}\, x + \arg\ \mathrm{cs}\, k'x = K, \tag{50}$$

$$\arg\ \mathrm{dc}\, x + \arg\ \mathrm{ns}\, x \ = K, \tag{51}$$

$$\arg\ \mathrm{nc}\, x + \arg\ \mathrm{ds}\, k'x = K. \tag{52}$$

The six corresponding expressions for K as a definite integral are

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2 . 1-k^2x^2)}}, \tag{53}$$

$$K = \int_0^1 \frac{dx}{\sqrt{(1-x^2 . k^2x^2+k'^2)}}, \tag{54}$$

$$K = \int_{k'}^1 \frac{dx}{\sqrt{(1-x^2 . x^2-k'^2)}}, \tag{55}$$

$$K = \int_0^\infty \frac{dx}{\sqrt{(x^2+1 . k'^2x^2+1)}}, \tag{56}$$

$$K = \int_1^\infty \frac{dx}{\sqrt{(x^2-1 . x^2-k^2)}}, \tag{57}$$

$$K = \int_1^\infty \frac{dx}{\sqrt{(x^2-1 . k'^2x^2+k^2)}}. \tag{58}$$

10. The denominators in the six different forms among the integrals (35)–(46) consist of all the combinations of two radicals of the forms

$$\sqrt{(a^2-x^2)}, \quad \sqrt{(x^2-a^2)} \text{ and } \sqrt{(x^2+a^2)}$$

where the two radicals may have the same form. As formulæ for integration it seems best to retain equations (35)–(37) and (41)–(43), which is the selection analogous to the choice of the forms arc sin x, arc tan x, and arc sec x in integrating radicals of the second degree.

11. Equations (47)–(52) obviously give the functions of the coamplitude $K-u$, thus the first three give

$$\operatorname{sn}(K-u) = \operatorname{cd} u,$$
$$\operatorname{cn}(K-u) = k' \operatorname{sd} u,$$
$$\operatorname{dn}(K-u) = k' \operatorname{nd} u,$$

and the others give consistent results, all of which may be expressed thus:

$$\text{if } u' = K-u, \quad s':c':d':n' = c:k's:k'n:d. \tag{59}$$

It is readily inferred that

$$\text{if } u' = K+u, \quad s':c':d':n' = c:-k's:k'n:d. \tag{60}$$

12. Let us now consider the functions of a pure imaginary quantity $u = iv$. Let $x = \operatorname{sn} u = \operatorname{sn} iv$, since the function sn is an odd function, x will be a pure imaginary quantity, say $x = iy$. Then, substituting in

$$u = \arg \operatorname{sn} x = \int_0 \frac{dx}{\sqrt{(1-x^2.1-k^2x^2)}},$$
$$iv = \int_0 \frac{i\,dy}{\sqrt{(1+y^2.1+k^2y^2)}};$$

or

$$v = \int_0 \frac{dy}{\sqrt{(1+y^2.1+k^2y^2)}}.$$

This value of v differs from $\arg \operatorname{sc} y$ [see equation (41)] only in the respect that k^2 takes the place of k'^2; hence

$$v = \arg \operatorname{sc}(y, k');$$

that is

$$y = \operatorname{sc}(v, k').$$

Thus

$$\operatorname{sn} iv = iy = i \operatorname{sc}(v, k'); \tag{61}$$

also, in a similar manner or directly from the quadratic relations, we find

$$\operatorname{cn} iv = \operatorname{nc}(v, k'), \tag{62}$$

and

$$\operatorname{dn} iv = \operatorname{dc}(v, k'). \tag{63}$$

13. Now, K' being defined as the same function of k' that K is of k, we have, putting $u = iv$, and n being an integer,

$$\operatorname{sn}(u + 4niK') = \operatorname{sn} i(v + 4nK') = i \operatorname{sc}(v + 4nK', k'),$$

by equation (61); but, since the functions to the modulus k' have the period $4K'$,

$$\operatorname{sc}(v + 4nK', k') = \operatorname{sc}(v, k');$$

hence

$$\operatorname{sn}(u + 4niK') = i \operatorname{sc}(v, k') = \operatorname{sn} iv = \operatorname{sn} u,$$

that is, $4iK'$ is a period of the function sn, and consequently also of the other elliptic functions.

Second Note on Weierstrass' Theory of Elliptic Functions.

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Origin of the Periodic Functions.

Instead of beginning his investigations by any reference to the elliptic integrals as did Legendre, Abel and Jacobi, Weierstrass seeks the origin of the periodic functions in a fundamental function-theoretical problem, namely: a series of argument values following a given analytical law inserted successively in a single valued analytic function of one variable gives rise to a series of corresponding functional values connected by a law which depends upon the character of the function when that is supposed given, but when the law connecting the argument values and a second law connecting the corresponding functional values are given, then the general character of the function is thereby determined, and, retaining either of the given laws, any class of functions whatever can be defined by a suitable choice of the other law. We may choose, for example, to let the successive argument-values form an arithmetic series, so that, u_1, u_2, u_3 being any three consecutive values, the analytical law connecting them is represented by $2u_2 = u_1 + u_3$. If now the corresponding functional values form a geometric series, we arrive easily at the exponential function as the solution, and if the two laws be exchanged we shall have defined, as might have been expected, the converse function or

$$\int_1 \frac{dx}{x}.$$

Retaining for the series of argument-values the above law $2u_2 = u_1 + u_3$, I shall show that, if the functional values are connected by a law expressible as a rational integral function, we shall have defined the single valued doubly-periodic functions. In other words Weierstrass starts from the addition-theorem.

Let $\phi(u)$ be a single valued analytic function possessing the last named property, it must, therefore, be developable in a series arranged according to the

positive and ascending powers of $u-a$, where a is not a singular point for ϕu. I shall call this a "power series of $(u-a)$," namely,

$$\phi u = A_0 + A_1(u-a) + A_2(u-a)^2 + \ldots$$

Substitute for u the values successively, u_1, u_2, u_3, where $2u_2 = u_1 + u_3$,

$$\phi(u_1) - A_0 = A_1(u_1 - a) + A_2(u_1 - a)^2 + \ldots$$
$$\phi(u_2) - A_0 = A_1(u_2 - a) + A_2(u_2 - a)^2 + \ldots$$
$$\phi(u_3) - A_0 = A_1(u_3 - a) + A_2(u_3 - a)^2 + \ldots$$

We can choose a so that $A_1 \gtrless 0$, and the series will converge and accurately represent the value of the function if only mod. $(u_1 - a)$ and mod. $(u_3 - a)$ be taken sufficiently small, since mod. $u_2 - a$ will then of necessity be within the circle of convergence. By inverting the series we can therefore develop $(u-a)$ in a power series of $\phi u - A_0$, which series will converge so long as mod. $(\phi u - A_0)$ be kept within corresponding limits. We are justified therefore in writing

$$u_1 - a = \frac{1}{A_1}(\phi u_1 - A_0) + \ldots$$
$$u_2 - a = \frac{1}{A_1}(\phi u_2 - A_0) + \ldots$$
$$u_3 - a = \frac{1}{A_1}(\phi u_3 - A_0) + \ldots$$

Since now $u_2 = \frac{u_1 + u_3}{2}$ we have

$$u_2 - a = \frac{1}{A_1}(\phi u_2 - A_0) + \ldots = \frac{1}{A_1}\left(\frac{\varphi u_1 + \varphi u_3}{2} - A_0\right) + \ldots$$

which establishes a relation between ϕu_1, ϕu_2, ϕu_3,

$$\text{or } F(\phi u_1, \phi u_2, \phi u_3) = 0.$$

I shall examine that class of functions where F represents a rational integral function. This addition equation can be shown in another form. So long as $u'_1 + u'_3 = u_1 + u_3$, then $u_2 = u'_2$ and the equation

$$F(\phi u'_1, \phi u_2, \phi u'_3) = 0$$

will also hold. Eliminating ϕu_2 we obtain

$$G(\phi u_1, \phi u_3, \phi u'_1, \phi u'_3) = 0.$$

Choosing u_1 and u_3 so that mod. $(u_1 + u_3)$ is within the limit of convergence, we can make $u'_3 = 0$, whence $u'_1 = u_1 + u_3$ and the last equation becomes

$$G(\phi u_1, \phi u_3, \phi(u_1 + u_3)) = 0.$$

By differentiating first with respect to ϕu_1 and then with respect to ϕu_3 we arrive at another form still of the addition equation. Writing for convenience instead of ϕu_1, ϕu_3, $\phi(u_1 + u_3)$; x, y, z, respectively, and instead of u_3, v, we have

$$G(x, y, z) = 0,$$

$$\frac{\partial G}{\partial x}\frac{\partial x}{du} + \frac{\partial G}{\partial z}\frac{dz}{du} = 0,$$

$$\frac{\partial G}{\partial y}\frac{dy}{dv} + \frac{\partial G}{\partial z}\frac{dz}{dv} = 0,$$

whence
$$\frac{\partial G}{\partial x}\frac{dx}{du} - \frac{\partial G}{\partial y}\frac{dy}{dv} = 0;$$

which last equation combined with the first gives

$$G_1\left(x,\ y,\ \frac{dx}{du},\ \frac{dy}{dv}\right) = 0,$$

which does not contain z. If now we assign to $v = u_3$ a constant value, the last equation yields an equation between x and $\frac{dx}{du}$, for if G_1 should vanish identically, that is if every coefficient of $x^\lambda\left(\frac{dx}{du}\right)^\mu$ should vanish, that would give a relation between y and $\frac{dy}{dv}$ which amounts to the same thing. The two equations

$$G(x,\ y,\ z) = 0,$$

$$\frac{\partial G}{\partial x}\frac{dx}{du} - \frac{\partial G}{\partial y}\frac{dy}{dv} = 0$$

must have at least one common root, therefore at least one common factor which is a linear function of z and a rational integral function of the coefficients of z in the two equations; therefore $z = R\left(x,\ y,\ \frac{dx}{du},\ \frac{dy}{dv}\right)$, that is to say, $\phi(u+v) = R(\phi u,\ \phi v,\ \phi' u,\ \phi' v)$, where R is a rational function.

If $\phi(u+v)$ be developable in a power series which converges for mod. $u < \frac{r}{2}$ and mod. $v < \frac{r}{2}$, then if we write $u = v$, the function

$$\phi(2u) = R(\phi u,\ \phi' u)$$

being a rational function is representable as the quotient of two integral functions, and

$$\phi u = R\left(\phi\frac{u}{2},\ \phi'\frac{u}{2}\right) = \frac{f_0\left(\phi\frac{u}{2},\ \phi'\frac{u}{2}\right)}{g_0\left(\phi\frac{u}{2},\ \phi'\frac{u}{v}\right)}$$

or
$$\phi u = \frac{f_1 u}{g_1 u},$$

where f_1 and g_1 converge for mod. $u < 2r$. In like manner

$$\phi\frac{u}{2} = \frac{f_1\frac{u}{2}}{g_1\frac{u}{2}} = \frac{f_2 u}{g_2 u},$$

where f_2 and g_2 converge for mod. $u < 4r$, and proceeding in this manner, the function ϕu can be represented as the quotient of two functions fu and gu with infinitely large circles of convergence. But since the quotient coincides with ϕu for all values of u such that mod. $u < r$, therefore it is the analytical continuation of ϕu, and represents the same function. Since ϕu is represented as the quotient of two power series which converge for all finite values of the argument, therefore ϕu has no essential singularities unless it be at infinity. These two functions, fu, gu, are either rational or transcendental integral functions. Whether they are rational or transcendental depends upon the number and kind of singular points which the function ϕu possesses. The case where both fu and gu are rational does not interest us. When fu is transcendental and gu rational then ϕu has one essential singularity at infinity and a finite number of non-essential singularities or poles. When, finally, both fu and gu are transcendental, the number of polar singularities is infinite, although there is still only a finite number of them in any finite region. When one or both of the functions fu, gu, are transcendental, then ϕu is periodic, for, being transcendental, the equation

$$\phi v - b = 0$$

is satisfied for as many different values of v as one pleases, for example, for $n+1$ values, so that $$\phi v_1 = \phi v_2 = \ldots \phi v_{n+1} = b.$$

Let the equation $$g(\phi(u+v),\ \phi u,\ b) = 0$$

be of the n^{th} degree in $\phi(u+v)$, it can then be satisfied for only n different values of $\phi(u+v)$, whereby the only condition is mod. $u < r$. But it is satisfied for $n+1$ different values of v; therefore for at least two values of v we must have

$$\phi(u+v_\lambda) = \phi(u+v_\mu),$$

or, writing $$v_\lambda - v_\mu = p,$$

$$\phi(u+p) = \phi(u),$$

and p is a period of the function. Any multiple of p is also a period. The smallest period then gives rise by its successive multiples to all the others of the class. If there is another class of periods, then they must be founded on a unit which is additively independent of the unit on which the first class is founded. But there are only two such units or rather, perhaps, unit-pairs. An analytic function of one variable cannot therefore have more than two classes of periods, because only two unit-pairs are possible so long as the fundamental laws of association and commutation hold. The Weierstrassian proof of this, which I venture to reproduce here, is as follows:

Let us suppose $e_1, e_2 \dots e_n$ to be n unit-pairs which are additively independent. We must also assume that the products of the units themselves are expressible as linear functions of the units. In order then that it may always be possible to carry out the operation of division, except when the divisor is zero, we should always be able to determine the coefficients of y in $xy = z$, when x and z are known and finite, and when

$$x = x_1 e_1 + x_2 e_2 + \dots x_n e_n$$
$$y = y_1 e_1 + y_2 e_2 + \dots y_n e_n$$
$$z = z_1 e_1 + z_2 e_2 + \dots z_n e_n$$
$$e_\lambda e_\mu = \alpha'_{\lambda\mu} e_1 + \alpha''_{\lambda\mu} e_2 + \dots \alpha^{(n)}_{\lambda\mu} e_n,$$

where $x_1, x_2 \dots x_n, y_1, y_2 \dots y_n, z_1, z_2 \dots z_n, \alpha'_{\lambda\mu}, \alpha''_{\lambda\mu} \dots \alpha^{(n)}_{\lambda\mu}$ are positive or negative real numbers. Owing to the independence of the unit-pairs as regards the notion of greater and less, we shall have, to determine the coefficients $y_1, y_2 \dots y_n$ of y, the n equations

$$(\alpha'_{11} x_1 + \alpha'_{21} x_2 + \dots \alpha'_{n1} x_n) y_1 + \dots + (\alpha'_{1n} x_1 + \alpha'_{2n} x_2 + \dots + \alpha'_{nn} x_n) y_n = z_1$$
$$(\alpha''_{11} x_1 + \alpha''_{21} x_2 + \dots \alpha''_{n1} x_n) y_1 + \dots + (\alpha''_{1n} x_1 + \alpha''_{2n} x_2 + \dots + \alpha''_{nn} x_n) y_n = z_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$(\alpha^{(n)}_{11} x_1 + \alpha^{(n)}_{21} x_2 + \dots \alpha^{(n)}_{n1} x_n) y_1 + \dots + (\alpha^{(n)}_{1n} x_1 + \alpha^{(n)}_{2n} x_2 + \dots + \alpha^{(n)}_{nn} x_n) y_n = z_n$$

It will now be found impossible to transform the determinant so that it will not vanish for any system of values of $x_1, x_2 \dots x_n$ except the one

$$x_1 = x_2 = \dots = x_n = 0$$

if the number of unit-pairs is greater than two. If however $n = 2$ the determinant $(\alpha'_{11} x_1 + \alpha'_{12} x_2)(\alpha''_{12} x_1 + \alpha''_{22} x_2) - (\alpha''_{11} x_1 + \alpha''_{12} x_2)(\alpha'_{12} x_1 + \alpha'_{22} x_2)$ must be and is expressible in the form

$$A x_1^2 + B x_2^2,$$

where A and B are positive. This determinant evidently does not vanish except for the single system of values $x_1 = x_2 = 0$.

Except therefore when the divisor is zero, the operation of division can always be carried out with finite numbers based on one or on two, but not more than two pairs of units, retaining the associative and commutative laws of arithmetic. Analytic functions of one variable therefore, which possess an addition equation and have an infinite number of polar singularities can be displayed as quotients of two power-series which converge for all finite arguments, and are periodic, with one and not more than two fundamental pairs of periods.

Fixing the attention upon the circumstance that the function will possess an infinite number of polar singularities, or what amounts to the same thing, an

infinite number of zero points, I shall proceed to determine a function by means of its zero points and by the fact of its convergence in general for all finite argument values.

ORIGIN OF THE SIGMA-FUNCTION.

If we seek a function gu such that

(1) $$g'_u = F\sum_\nu c_\nu u^\nu,$$

(2) $$g(0) = c,$$

the conditions can always be satisfied when the power-series $\sum_\nu c_\nu u^\nu$ converges, for then the series $\sum_\nu \frac{c_\nu}{\nu+1} u^{\nu+1}$ converges also, and both sides of (1) have the same radius of convergence. The function is easily seen to be $ce^{g(u)}$, where $g(0) = 0$, and gu means an entire function. But the following conditions will now be imposed in addition to the above. The function (3) shall vanish for $u = a_\nu$, $\nu = 0 \pm 1 \ldots \pm \infty$, with the condition lim. $a_\nu = \infty$, $(\nu = \infty)$, that is, the number of a's is infinite, but there shall be only a finite number of them in a finite region.

(4) $$G(b_\nu) \gtrless 0, \quad b_\nu \gtrless a_\nu,$$

(5) the function shall be developable in a power-series which converges for all finite values of the argument.

The last condition determines

$$Gu = C_\lambda(u-a)^\lambda + C_{\lambda+1}(u-a)^{\lambda+1} + \ldots$$

whence $$G'u = \lambda C_\lambda(u-a)^{\lambda-1} + (\lambda+1)\, C_{\lambda+1}(u-a)^{\lambda-1} + \ldots$$

and

$$\frac{G'}{G} = \frac{\lambda + \frac{\lambda+1}{\lambda} C_\lambda(u-a) + \ldots}{(u-a)\left(1 + \frac{C_{\lambda+1}}{C_\lambda}(u-a) + \ldots\right)}$$

$$= \frac{\lambda}{u-a} + \Sigma(u-a)$$

where $\Sigma(u-a)$ means a power-series of $(u-a)$. The first logarithmic derivative of the function will therefore be essentially of the form

$$Fu = \frac{G'}{G} = \sum_\nu \frac{1}{u-a_\nu},$$

or more generally $$\Sigma \frac{g_\nu u}{(u-a_\nu)\, g_\nu(a_\nu)},$$

which I shall specialize somewhat and write

$$Fu = \frac{G'}{G} = \Sigma \frac{u^{m_\nu}}{(u-a_\nu)\, a_\nu^{m_\nu}},$$

where to the series $a_1, a_2 \ldots a_n$ can be coordinated a second series $m_1, m_2 \ldots m_n$ such that Fu will converge for all values of the argument. I shall now form a transcendental function with the same infinite points as Fu. Taking first a single term of Fu

$$\frac{u^m}{(u-a)\,a^m}$$

this can be written in the form

$$\frac{1}{u-a}+\frac{u^m-a^m}{(u-a)\,a^m}=\frac{1}{u-a}+\frac{u^{m-1}}{a^m}+\frac{u^{m-2}}{a^{m-1}}+\ldots\frac{u^0}{a}.$$

If now we write

$$fu=\left(1-\frac{u}{a}\right)e^{\frac{u}{a}+\frac{1}{2}\frac{u^2}{a^2}+\ldots\frac{1}{m}\frac{u^m}{a^m}},$$

we shall have

$$\frac{f'u}{fu}=\frac{1}{u-a}+\frac{1}{a}+\frac{u}{a^2}+\ldots+\frac{u^{m-1}}{a^m}=\frac{u^m}{(u-a)\,a^m},$$

and the general form of the function is discovered. In order to include all the terms we will denote by

$E(u, 0)$ the function $1-u$

$E(u, 1)$ " " $(1-u)e^u$

$E(u, 2)$ " " $(1-u)e^{u+\frac{1}{2}u^2}$

. . . .

$E(u, n)$ " " $(1-u)e^{u+\frac{1}{2}u^2+\ldots+\frac{1}{n}u^n}$

whereupon the m^{th} term of Fu is

$$\frac{u^m}{(u-a)\,a^m}=\frac{E'\left(\frac{u}{a},\, m\right)}{E\left(\frac{u}{a},\, m\right)}=\frac{d}{du}\log E\left(\frac{u}{a},\, m\right)$$

and

$$Fu=\sum_{\nu=0}^{\infty}\frac{u^{m_\nu}}{(u-a_\nu)\,a_\nu^{m_\nu}}=\frac{d}{du}\log\prod_{\nu=0}^{\infty}E\left(\frac{u}{a_\nu},\, m_\nu\right)=\frac{G'}{G}.$$

If zero is to be a zero point for G of the order λ, since we dare not include this value among the a's, we must write

$$G=u^\lambda\prod_{\nu=0}^{\infty}E\left(\frac{u}{a_\nu},\, m_\nu\right).$$

The conditions of convergence are best discovered by separating the sum for $F(u)$ into two parts

$$Fu=\frac{G'}{G}=\sum_{\nu=0}^{m}\frac{u^{m_\nu}}{(u-a_\nu)\,a_\nu^{m_\nu}}+\sum_{m+1}^{+\infty}\frac{u^{m_\nu}}{(u-a_\nu)\,a_\nu^{m_\nu}},$$

and throwing the second sum into the form

$$-\sum_{\nu=m+1}^{\infty}\frac{1}{1-\frac{u}{a_\nu}}\cdot\frac{u^{m_\nu}}{a_\nu^{m_\nu+1}}=-\sum_{\nu=m+1}^{\infty}\sum_{r=0}^{\infty}\frac{u^{m_\nu+r}}{a_\nu^{m_\nu+r+1}};$$

whereupon
$$G = u^{\lambda} \prod_{\nu=0}^{m} E\left(\frac{u}{a_\nu}, m_\nu\right) - \sum_{\nu=m+1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{m_\nu + r} \frac{u^{m_\nu + r}}{a_\nu^{m_\nu + r}}.$$

If now the sequence of zero-points a_ν forms only a single arithmetic series, then the uniform convergence will be assured if we assume $m_1 = m_2 = \ldots = m_n = 1$, when
$$G = u^{\lambda} \prod_{\nu=0}^{\infty} \left(1 - \frac{u}{a_\nu}\right) e^{\frac{u}{a_\nu}}.$$

If the sequence of zero values extends from $-\infty$ to $+\infty$, then by multiplying together corresponding positive and negative pairs, and when $\lambda = 1$, $a_\nu = n\pi$,
$$G = u \prod_{n=-\infty}^{+\infty}{}' \left(1 - \frac{u}{n\pi}\right) e^{\frac{u}{n\pi}} = u \prod_{n=+1}^{+\infty} \left(1 - \frac{u^2}{(n\pi)^2}\right).$$
This last formula has indeed long been known, but the first form where the sine is represented as the product of "prime-functions" is the discovery of Weierstrass. In case the a_ν's form two additively independent series of zero points so that for example
$$a_\nu = w = m2\omega + m'2\omega'.$$
$$m, m' = 0 \pm 1 \pm 2 \ldots \pm \infty,$$
the formula becomes
$$G = u \prod_{\nu=-\infty}^{+\infty} E\left(\frac{u}{w}, m_\nu\right),$$
where m_ν is so to be determined that the function shall converge uniformly for all finite argument values. The convergence depends on the convergence of
$$\sum_{\nu=m+1}^{\infty} \sum_{r=1}^{\infty} \frac{1}{m_\nu + r} \frac{u^{m_\nu} + r}{a_\nu^{m_\nu} + r}.$$
To this end it is sufficient if the convergence of
$$\sum_{\nu=m+1}^{\infty} \frac{1}{m_\nu} \cdot \text{mod.} \frac{1}{a_\nu} \cdot \text{mod.} \left(\frac{u}{a_\nu}\right)^{m_\nu}$$
is assured. Since now a_ν is a complex number it is no longer the case that $\sum_\nu \frac{1}{a_\nu^2}$ converges. It can be shown however that $\Sigma \frac{1}{a_\nu^{m_\nu + 1}}$ where a_ν is complex does converge for $m_\nu = 2$ or more. It is sufficient then to take $m_\nu = 2$ and the function is
$$G = u \prod_{-\infty}^{+\infty}{}' \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}} = \sigma u$$
$$w = m2\omega + m'2\omega'$$
$$m_1 m_1 = 0 \pm 1 \pm 2 \ldots \pm \infty$$
$$\text{except } m = m' = 0,$$

and the required function is displayed as the continued product of an infinite number of prime functions, each of which has its own polar singularity and each of which has one essential singularity at infinity.

The function σu is the simplest possible function that satisfies the condition named. Being produced in the same manner as the sine, the sigma function degrades into the sine easily at the limit,

$$\lim. R\left(\frac{\omega'}{\omega i}\right) = \infty \quad (R \text{ signifies real component}) \text{ when}$$

$$\lim. \sigma u = u\Pi'\left(1 - \frac{u}{m2\omega}\right) e^{\frac{u}{m2\omega} + \frac{1}{2}\left(\frac{u}{m2\omega}\right)^2}$$

or taking the second form (see p. 178, vol. VI),

$$\lim. \sigma u = u\Pi'\left(1 - \left(\frac{u}{m2\omega}\right)^2\right) e^{\left(\frac{u}{m2\omega}\right)^2}$$

$$= \frac{2\omega}{\pi} e^{\frac{1}{6}\left(\frac{u\pi}{2\omega}\right)^2} . \sin \frac{u\pi}{2\omega},$$

where the zero-points of lim. σu all lie on the real axis, and are contained in the formula $\quad n . 2\omega, \; n = 0 \pm 1, \; \pm 2, \; \pm \infty.$

A similar formula applies to a function with the zero-points represented by each of the formulæ $\quad n . 2\omega \pm 2\omega', \; n2\omega \pm 2 . 2\omega' \ldots$

or generally $\quad n2\omega + n' 2\omega'.$

Mnltiplying together at first those pairs which are symmetrically situated with respect to the origin, we arrive at an expression of the sigma-function as a simply infinite product

$$\sigma u = \frac{2\omega}{\pi} \sin \frac{u\pi}{2\omega} e^{\frac{1}{6}\left(\frac{u\pi}{2\omega}\right)^2} \Pi_n \left\{ \frac{\sin \frac{\pi}{2\omega}(2n\omega' - u) \sin \frac{\pi}{2\omega}(2n\omega' + u)}{\sin^2 \frac{n\omega'\pi}{\omega}} . e^{\frac{1}{\sin^2 \frac{n\omega'\pi}{\omega}} \cdot \left(\frac{u\pi}{2\omega}\right)^2} \right\}.$$

By writing

$$\eta = \frac{\pi^2}{2\omega}\left(\frac{1}{6} + \sum_n \frac{1}{\sin^2 \frac{n\omega'\pi}{\omega}}\right),$$

$$\sigma u = e^{\frac{\eta u^2}{2\omega}} \cdot \frac{2\omega}{\pi} \sin \frac{u\pi}{2\omega} \cdot \Pi_n \left(1 - \frac{\sin^2 \frac{u\pi}{2\omega}}{\sin^2 \frac{n\omega'\pi}{\omega}}\right).$$

The first, I believe, to discover that the elliptic functions could be got at by starting from the infinite product

$$\Pi\left(1 - \frac{x}{n}\right)$$

was Eisenstein in *Crelle's Journal*, Bd. XXVII, where he has two articles on this subject. He remarks that the quotient of two infinite products of the form

$$\Pi\left(1-\frac{z}{\lambda A+\lambda' A'}\right),$$

where $\lambda=2,\ 4,\ 6\ldots,\ \lambda'=1,\ 3,\ 5\ldots,$ gives rise to the elliptic functions. In *Crelle*, Bd. XXXV, he gives a long discussion also on the convergence of these last products. The necessity of a factor of convergence seems not to have occurred to him. He remarks that the easiest approach to the study of elliptic functions is through infinite products, they showing most plainly the periods and *also the zero and infinite points* of the function, and yet the notion of prime factors escaped him, and was first brought out by Weierstrass thirty years later. With the aid of the expression for the sine

$$\frac{\sin \pi x}{\pi}=x\Pi\left(1-\frac{x}{n}\right)e^{\frac{x}{n}},\ n=\pm 1,\ \pm 2\ldots\pm\infty$$

$$=x\Pi\left(1-\frac{x^2}{n^2}\right) \qquad n=1,\ 2\ldots+\infty$$

and the identity $$1-\frac{x-a}{b}=\left(1+\frac{a}{b}\right)\left(1-\frac{x}{a+b}\right),$$

we arrive at the above expressions of σu as a singly infinite product of sines.

The function σu has therefore been determined by means of its zero-points and the condition that its only essential singularity is at infinity. Its zero-points are contained in the formula

$$w=m2\omega=m'\,2\omega',$$

$$m,\ m'=0\pm 1\pm 2\ldots\pm\infty.$$

If then a be a non-singular argument value for the function σu, the latter is developable in the vicinage of a and of all values congruent to a in the form

$$\sigma u=(u-a)\{A_0+A_1(u-a)+\ldots\}.$$

If, farther, ϕu be a doubly periodic function with the zero-points

$$a_1,\ a_2,\ a_3,\ \ldots a_r$$

and the (polar) infinite points

$$b_1,\ b_2,\ b_3,\ \ldots b_s$$

within each of its period parallelograms, we can then form the quotient

$$P=\frac{\sigma(u-a_1)\sigma(u-a_2)\ldots\sigma(u-a_r)}{\sigma(u-b_1)\sigma(u-b_2)\ldots\sigma(u-b_s)},$$

which has the same zero and infinite points as ϕu. If we divide ϕu by P, we shall have a function which has no zero nor infinite points except at infinity, and

is therefore of the form $e^{G(x)}$ where $G(x)$ is an entire function. The function ϕu must therefore have the form

$$\phi u = \frac{\sigma(u-a_1)\,\sigma(u-a_2)\ldots\sigma(u-a_r)}{\sigma(u-b_1)\,\sigma(u-b_2)\ldots\sigma(u-b_s)}\cdot e^{G(u)}.$$

It remains to be shown, however, under what conditions such a sigma-quotient represents a doubly periodic function; in particular, to determine the form of the function Gu, and the relations connecting the a's and b's. We have

$$\sigma(u-a_\mu+2\omega) = -\,\sigma(u-a_\mu)\,e^{2\eta(u-a_\mu+\omega)},$$

the proof of which I beg to omit at present, as it would extend beyond the intended limits of this article. Therefore

$$\phi(u+2\omega) = \phi u.(-1)^{r-s}.e^{2(r-s)\eta(u+\omega)-2\eta\{\Sigma a_\nu-\Sigma b_\nu\}+G(u+2\omega)-G(u)},$$

and, in order that 2ω may be a period of ϕu, the exponent of e must be a multiple of πi. We may write therefore

$$G(u+2\omega) - Gu = 2\eta\,\{\Sigma a_\nu - \Sigma b_\nu - (r-s)(u+\omega)\} + k\pi i,$$

where k being an integer must be constant in order that G may be a continuous function. In like manner

$$G(u+2\omega') - Gu = 2\eta'\{\Sigma a_\nu - \Sigma b_\nu - (r-s)(u+\omega')\} + k'\pi i.$$

Differentiating twice,
$$G''(u+2\omega) - G''u = 0$$
$$G''(u+2\omega') - G''u = 0.$$

The function G'' must therefore be a constant, or else an exponential series, but this last it cannot be and be doubly periodic as the equations show, G is then an integral function of a degree not higher than the second; or

$$Gu = \alpha u^2 + \beta u + \gamma,$$
$$G(u+2\omega) = Gu + 4\alpha\omega\,(u+\omega) + 2\beta\omega$$
$$G(u+2\omega') = Gu + 4\alpha\omega'(u+\omega') + 2\beta\omega',$$
$$2\eta\{\Sigma a_\nu - \Sigma c_\nu - (r-s)(u+\omega)\} + k\pi i = 4\alpha\omega\,(u+\omega) + 2\beta\omega$$
$$2\eta'\{\Sigma a_\nu - \Sigma b_\nu - (r-s)(u+\omega')\} + k'\pi i = 4\alpha\omega'(u+\omega') + \alpha\beta\omega',$$

whence
$$4\alpha\omega + 2\eta\,(r-s) = 0,$$
$$4\alpha\omega' + 2\eta'(r-s) = 0.$$

Since however
$$\omega\eta' - \omega'\eta = \pm\frac{\pi i}{2} \gtrless 0$$

which is shown from

$$\sigma(u+2\omega+2\omega') = -\,e^{2\eta(u+\omega+2\omega')}.\sigma(u+2\omega') = \sigma u.e^{2\eta(u+\omega+2\omega')+2\eta'(u+\omega')}$$

and
$$\sigma(u+2\omega+2\omega') = -\,e^{2\eta'(u+\omega'+2\omega)}.\sigma(u+2\omega) = \sigma u.e^{2\eta(u+\omega'+2\omega)+2\eta(u+\omega)},$$

we must have
$$r - s = 0,$$

whence
$$\alpha = 0,\quad r = s,\quad Gu = \beta u + \gamma,$$

and the exponent becomes

$$2\eta\{\Sigma a_\nu - \Sigma b_\nu\} - 2\beta\omega = 2k\pi i,$$

where $2k$ must be written instead of k, since now $r = s$, and therefore from

$$\phi(u + 2\omega) = \phi u . e^{2\eta\{\Sigma a_\nu - \Sigma b_\nu\} - 2\beta\omega},$$

in order that 2ω may be a period of ϕu, the exponent of e must be a multiple of $2\pi i$ instead of πi. Whence, writing

$$\Sigma a_\nu - \Sigma b_\nu = D,$$

we have

$$\eta D + k\pi i = \beta\omega,$$
$$\eta' D + k'\pi i = \beta\omega',$$
$$(\eta\omega' - \eta'\omega) D = (k\omega' - k'\omega)\pi i$$
$$(\eta\omega' - \eta'\omega)\beta = (k'\eta - k\eta')\pi i$$

or since

$$\eta\omega' - \eta'\omega = \pm\frac{\pi i}{2}$$
$$D = \pm 2(k\omega' - k'\omega)$$
$$\beta = \pm 2(k\eta' - k'\eta)$$

and the function ϕ becomes

$$\phi(u) = C\frac{\sigma(u - a_1)\ldots\sigma(u - a_r)}{\sigma(u - b_1)\ldots\sigma(u - a_r)} \cdot e^{(2k'\eta - 2k\eta')u},$$

where k, k' are integers, and $\eta = \frac{\sigma'\omega}{\sigma\omega}$, $\eta' = \frac{\sigma'\omega'}{\sigma\omega'}$. If now instead of any one of the values a_ν and b_ν we take one greater or less by 2ω or $2\omega'$, the exponential factor will be increased by $ce^{\pm 2\eta u}$ or $ce^{\pm 2\eta' u}$, whereby $\Sigma a_\nu - \Sigma b_\nu$ still remains a multiple of the periods and k or k' will be increased or diminished by one. We can therefore always choose such values conjugate to a_ν and b_ν that $k = k' = 0$ and $\Sigma a_\nu = \Sigma b_\nu$, and there remains as the general expression of a doubly periodic function by means of sigma quotients,

$$\phi u = C\frac{\sigma(u - a_1)\ldots\sigma(u - a_r)}{\sigma(u - b_1)\ldots\sigma(u - b_r)},$$

where

$$\Sigma a_\nu = \Sigma b_\nu.$$

As regards the degree of ϕu in $\sigma(u - a)$ it is at once evident that it must be higher than the first, for if we had

$$\phi u = C\frac{\sigma(u - a_1)}{\sigma(u - b_1)},$$

since $\Sigma a_1 = \Sigma b_1$, that is, $a_1 = b_1$, the expression would reduce to a constant. The simplest form possible is therefore

$$\phi u = C\frac{\sigma(u - a_1)\sigma(u - a_2)}{\sigma(u - b_1)\sigma(u - b_2)},$$
$$a_1 + a_2 = b_1 + b_2,$$

which form contains those functions commonly called elliptic. This form must be discussed a little more fully, for from it springs perhaps the most convenient if not the most important formula in this system. If we write $b_1 = b_2 = 0$, and $a_1 = v$, the formula becomes

$$\phi u = C \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u},$$

where v is a constant upon which C evidently depends. We are at liberty to choose C so that

$$C = \frac{1}{\sigma^2 v},$$

whereupon

$$\phi u = \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v}$$

is a doubly periodic function of either u or v, the other assuming a non-singular constant value. Now the definition of $\wp u$ in my first note was

$$\begin{aligned}\wp u &= -\frac{d^2}{du^2}\log \sigma u \\ &= \frac{1}{u^2} + \Sigma' \left\{ \left(\frac{1}{u-w}\right)^2 - \frac{1}{w^2} \right\}\end{aligned}$$

where the value $w = 0$ was omitted. Differentiating,

$$\wp' u = -\frac{2}{u^3} - 2\Sigma' \left(\frac{1}{u-w}\right)^3,$$

or including the value $w = 0$ under the Σ,

$$\wp' u = -2\Sigma \left(\frac{1}{u-w}\right)^3,$$

which is evidently periodic, and

$$\begin{aligned}\wp'(u+2\omega) &= \wp' u \\ \wp'(u+2\omega') &= \wp' u,\end{aligned}$$

and on integrating

$$\begin{aligned}\wp(u+2\omega) &= \wp u + C, \\ \wp(u+2\omega') &= \wp u + C'.\end{aligned}$$

But

$$\begin{aligned}\sigma(-u) &= -\sigma'(u), \\ \sigma'(-u) &= \sigma'(u),\end{aligned}$$

consequently

$$\frac{\sigma'(-u)}{\sigma(-u)} = -\frac{\sigma' u}{\sigma u}$$

and

$$\wp(-u) = \wp u,$$

whence

$$C = C' = 0,$$

and $\wp u$ is periodic. We have also

$$\wp u - \wp v = \Sigma \left\{ \left(\frac{1}{u-w}\right)^2 - \left(\frac{1}{v-w}\right)^2 \right\}.$$

The function $\wp u$ becomes infinite only for $u=0$ and for the congruent values. So long then as v is not given the value zero or any value congruent to zero, $\wp u - \wp v$ regarded as a function of u will have the same infinite points as $\wp u$. The zero points are $u = \pm v$, and the points congruent to these. It follows that

$$\wp u - \wp v = C \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u}$$

as a function of u alone, or

$$\wp u - \wp v = C' \frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v}$$

as a function of u or v, where C' is independent of both u and v. In order to determine C', develop both sides of the equation according to powers of u and v. As given in my former note, p. 178,

$$\wp u = \frac{1}{u^2} + \ldots$$

$$\sigma u = u + \ldots,$$

so that

$$\wp u - \wp v = \frac{1}{u^2} - \frac{1}{v^2} + \ldots$$

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v} = \frac{1}{v^2} - \frac{1}{u^2} + \ldots,$$

and we must have $C' = -1$, or

$$\wp u - \wp v = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v} = -\begin{vmatrix} 1 & \wp u \\ 1 & \wp v \end{vmatrix},$$

which is the "pocket edition" of the elliptic functions mentioned.

But we can go farther and express through sigma-quotients such expressions as

$$\begin{vmatrix} 1 & \wp u & \wp' u \\ 1 & \wp v & \wp' v \\ 1 & \wp w & \wp' w \end{vmatrix}.$$

The infinite points (poles) of $\wp u$ and $\wp' u$ are congruent to zero, $\wp u$ becoming infinite of the second and $\wp' u$ of the third order for $u = 0 \pm w$. The determinant must therefore as a linear function of $\wp$ and $\wp'$ have also a three-fold infinite point at zero and the congruent points, therefore also three zero points to which all other vanishing points are congruent. The determinant vanishes evidently for $u = v$, $u = w$, $u = -(v+w)$, and can be expressed as follows:

$$C \frac{\sigma(u+v+w)\sigma(u-v)\sigma(v-w)}{\sigma^3 u} \times$$

$$C_1 \frac{\sigma(v+w+u)\sigma(v-w)\sigma(v-u)}{\sigma^3 v} \times$$

$$C_2 \frac{\sigma(w+u+v)\sigma(w-u)\sigma(w-v)}{\sigma^3 w},$$

where C depends on v and w, C_1 on w and u, C_2 on u and v. We may write then for the determinant

$$C' \frac{\sigma(u+v+w)\sigma(u-v)\sigma(u-w)\sigma(v-w)}{\sigma^3 u . \sigma^3 v . \sigma^3 w} = \begin{vmatrix} 1 & \wp u & \wp' u \\ 1 & \wp v & \wp' v \\ 1 & \wp w & \wp' w \end{vmatrix}$$

where C' now depends on u, v and w.

Multiplying by u^3 and then making $u = 0$, and remembering that

$$\wp' u = \frac{-2}{u^3} + \ldots, \qquad C' \frac{\sigma(v+w)\sigma(v-w)}{\sigma^2 v \sigma^2 w} = -2 \begin{vmatrix} 1 & \wp v \\ 1 & \wp w \end{vmatrix}$$

whence, comparing with the foregoing, $\quad C' = -2$, and

$$\frac{\sigma(u+v+w)\sigma(u-v)\sigma(v-w)}{\sigma^3 u \sigma^3 v \sigma^3 w} = -\frac{1}{2} \begin{vmatrix} 1 & \wp u & \wp' u \\ 1 & \wp v & \wp' v \\ 1 & \wp w & \wp' w \end{vmatrix},$$

and in a similar manner the determinants of higher degrees would be represented through sigma-quotients.

I will now deduce another relation which was merely dogmatically given in my first paper. Since $\wp' u$ is periodic and an odd function, we have

$$\wp'(u + \omega) = \wp'(u - \omega) = -\wp'(\omega - u),$$

and putting $u = 0$ $\qquad \wp'\omega = -\wp'\omega = 0,$

and likewise $\qquad \wp'\omega' = 0.$

If we set now in the above determinant of the third order, $v = \omega$ and $w = \omega'$, it becomes

$$\begin{vmatrix} 1 & \wp u & \wp' u \\ 1 & \wp\omega & 0 \\ 1 & \wp\omega' & 0 \end{vmatrix} = \wp' u \begin{vmatrix} 1 & \wp\omega \\ 1 & \wp\omega' \end{vmatrix}$$

and introducing the same changes in the left-hand side of the same equation we have at once

$$\wp' u = -2 \frac{\sigma(u+\omega+\omega')\sigma(u-\omega)\sigma(u-\omega')\sigma(\omega-\omega')}{\sigma^3 u . \sigma^3 \omega . \sigma^3 \omega'} \cdot \frac{1}{\begin{vmatrix} 1 & \wp\omega \\ 1 & \wp\omega' \end{vmatrix}}.$$

But

$$\begin{vmatrix} 1 & \wp\omega \\ 1 & \wp\omega' \end{vmatrix} = \frac{\sigma(\omega+\omega')\sigma(\omega-\omega')}{\sigma^3 \omega \sigma^3 \omega'},$$

whence

$$\wp' u = -\wp'(-u) = -2 \frac{\sigma(u+\omega+\omega')\sigma(u-\omega)\sigma(u-\omega')}{\sigma^3 u . \sigma\omega . \sigma\omega' . \sigma(\omega+\omega')},$$

$$(\wp' u)^2 = 4 \frac{\sigma(u+\omega+\omega')\sigma(\omega+\omega'-u)}{\sigma^2 u . \sigma^2(\omega+\omega')} \cdot \frac{\sigma(u+\omega)\sigma(u-\omega)}{\sigma^2 u . \sigma^2 \omega} \cdot \frac{\sigma(u+\omega')\sigma(u-\omega')}{\sigma^2 u . \sigma^2 \omega'},$$

that is, $\qquad = 4\big(\wp u - \wp(\omega + \omega')\big)(\wp u - \wp\omega)(\wp u - \wp\omega'),$

and, writing $\wp\omega = e_1$, $\wp(\omega + \omega') = e_2$, $\wp\omega' = e_3$, this becomes

$$(\wp' u)^2 = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3).$$

On writing now s for $\wp u$, this gives us of course the well-known Weierstrassian form of the elliptic integral,

$$u = \int \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}.$$

The function $(\wp' u)^2$ is thus expressed as a rational function of $\wp u$. From the definition

$$\wp u = -\frac{d^2}{du} \log \sigma u = \frac{1}{u^2} + \Sigma' \left(\frac{1}{(u-w)^2} - \frac{1}{w^2} \right).$$

But developing according to powers of u,

$$\frac{1}{u-w} = \frac{1}{w}\left(1 + \frac{u}{w} + \frac{u^2}{w^2} + \ldots\right)$$

$$\frac{1}{(u-w)^2} = \frac{1}{w^2} + 2\frac{u}{w^3} + 3\frac{u^2}{w^4} + 4\frac{u^3}{w^5} + \ldots$$

$\wp u = \frac{1}{u^2} + 2u\Sigma \frac{1}{w^3} + 3u^2\Sigma \frac{1}{w^4} + \ldots$, and writing $\Sigma \frac{1}{w^n} = c_n$, remarking that for n odd the sum vanishes, we have

$$\wp u = \frac{1}{u^2} + 3c_4 u^2 + 5c_6 u^4 + \ldots$$

$$(\wp u)^2 = \frac{1}{u^4}(1 + 2.3.c_4 u^4 + 10c_6 u^6 + \ldots)$$

$$(\wp u)^3 = \frac{1}{u^6}(1 + 9c_4 u^4 + 15c_6 u^6 + \ldots)$$

$$\wp' u = -\frac{2}{u_3} + 2.3.c_4 u + 4.5c_6 u^3 + 6.7.c_8 u^5 + \ldots$$

$$(\wp' u)^2 = \frac{1}{u_6}(4 - 24c_4 u^4 - 80c_6 u^6 + \ldots$$

whence the expression for $(\wp' u)^2$ takes the form

$$(\wp' u)^2 = 4\wp^3 u - 60c_4 \wp u - 140c_6$$
$$= 4\wp^3 u - g_1 \wp^2 u - g_2 \wp u - g_3,$$

where

$$g_1 = e_1 + e_2 + e_3 = 0$$

$$g_2 = -4.(e_2 e_3 + e_3 e_1 + e_1 e_2) = 2^2.3.5\Sigma \frac{1}{w^4}$$

$$g_3 = 4.e_1.e_2.e_3 = 2^2.5.7.\Sigma \frac{1}{w^6}.$$

The addition theorem for $\wp u$ may be derived from the pocket edition

$$\wp u - \wp v = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v}$$

by logarithmic differentiation, which gives

$$\frac{\sigma'(u+v)}{\sigma(u+v)} + \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' u}{\sigma u} = \frac{\wp' u}{\wp u - \wp v},$$

$$\frac{\sigma'(u+v)}{\sigma(u+v)} - \frac{\sigma'(u-v)}{\sigma(u-v)} - 2\frac{\sigma' v}{\sigma v} = \frac{-\wp' v}{\wp u - \wp v},$$

from which, as the addition theorem for the function $\frac{\sigma' u}{\sigma u}$,

$$\frac{\sigma'(u \pm v)}{\sigma(u \pm v)} = \frac{\sigma' u}{\sigma u} \pm \frac{\sigma' v}{\sigma v} + \frac{1}{2}\,\frac{\wp' u \mp \wp' v}{\wp u - \wp v}.$$

It is apparent that one more differentiation gives the addition theorem for $\wp u$, which can then easily be brought into the form given in the first paper,

$$\wp(u \pm v) = \frac{1}{4}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]^2 - \wp u - \wp v.$$

Weierstrass deduces the same theorem in another way which, as being characteristic, I reproduce. Equating to zero the above determinant

$$\begin{vmatrix} 1 & \wp u & \wp' u \\ 1 & \wp v & \wp' v \\ 1 & \wp w & \wp' w \end{vmatrix} = \wp' w(\wp v - \wp u) + \wp' v(\wp u - \wp w) + \wp' u(\wp w - \wp v) = 0,$$

we have on squaring

$$(\wp' w)^2(\wp u - \wp v)^2 - \{\wp' v(\wp u - \wp w) - \wp' u(\wp v - \wp w)\}^2 = 0,$$

or substituting the value of $(\wp' w)^2$ obtained above

$$(\wp u - \wp v)^2(4\wp^3 w - g_2 \wp w - g_3) - \{\wp' v\, \wp u - \wp' u\, \wp v - \wp w(\wp' v - \wp' u)\}^2 = 0$$

which is satisfied for $w = u,\ v,\ u + v$, that is for $\wp w = \wp u,\ \wp v,\ \wp(u + v)$.
Forming now the equation

$$4(\wp u - \wp v)^2\{(s - \wp u)(s - \wp v)(s - \wp(u + v))\} = 0,$$

this is likewise satisfied for $s = \wp u,\ \wp v,\ \wp(u + v)$,
and, since the coefficients of the highest powers, $(\wp w)^3$ and s^3, are the same in each, therefore all the coefficients of the same powers of $\wp w$ and s are equal. Equating those of $(\wp w)^2$ and s^2, we have without any reduction

$$-(\wp' u - \wp' v)^2 = 4(\wp u - \wp v)^2\{-\wp u - \wp v - \wp(u - v)\},$$

whence immediately

$$\wp(u + v) = \frac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2 - \wp u - \wp v,$$

as the addition theorem for the function $\wp u$.

Lectures on the Principles of Universal Algebra.

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LECTURE I.

PRELIMINARY CONCEPTIONS AND DEFINITIONS.

Apotheosis of Algebraical Quantity.

A matrix of a quadrate form historically takes its rise in the notion of a linear substitution performed upon a system of variables or carriers; regarded apart from the determinant which it may be and at one time was almost exclusively used to represent, it becomes an empty *schema* of operation, but in conformity with Hegel's principle that the Negative is the course through which thought arrives at another and a fuller positive, only for a moment loses the attribute of quantity to emerge again as quantity, if it be allowed that that term is properly applied to whatever is the subject of functional operation, of a higher and unthought of kind, and so to say, in a glorified shape,—as an organism composed of discrete parts, but having an essential and undivisible unity as a whole of its own. *Naturam expellas furcâ, tamen usque recurret.** The conception of multiple quantity thus rises upon the field of vision.

At first undifferentiated from their content, matrices came to be regarded as susceptible of being multiplied together; the word multiplication, strictly applicable at that stage of evolution to the content alone, getting transferred by a fortunate confusion of language to the schema, and superseding, to some extent, the use of the more appropriate word composition applied to the reiteration of substitution in the Theory of Numbers. Thus there came into view a process of multiplication which the mind, almost at a glance, is able to recognize must be subject to the associative law of ordinary multiplication, although not so to the

* *Chassez le naturel, il revient au galop*, a familiar quotation which I thought was from Boileau, but my friend Prof. Rabillon informs me is from a comedy of Destouches (born in 1680, died 1754).

commutative law; but the full significance of this fact lay hidden until the subject-matter of such operations had dropped its provisional mantle, its aspect as a mere schema, and stood revealed as bona-fide multiple quantity subject to all the affections and lending itself to all the operations of ordinary numerical quantity. This revolution was effected by a forcible injection into the subject of the concept of addition, *i. e.* by choosing to regard matrices as susceptible of being added to one another; a notion, as it seems to me, quite foreign to the idea of substitution, the *nidus* in which that of multiple quantity was laid, hatched and reared. This step was, as far as I know, first made by Cayley in his Memoir on Matrices, in the *Phil. Trans.* 1858, wherein he may be said to have laid the foundation-stone of the science of multiple quantity. That memoir indeed (it seems to me) may with truth be affirmed to have ushered in the reign of Algebra the 2d; just as Algebra the 1st, in its character, not as mere art or mystery, but as a science and philosophy, took its rise in Harriot's "Artis Analyticae Praxis," published in 1631, ten years after his death, and exactly 250 years before I gave the first course of lectures ever delivered on Multinomial Quantity, in 1881, at the Johns Hopkins University. Much as I owe in the way of fruitful suggestion to Cayley's immortal memoir, the idea of subjecting matrices to the additive process and of their consequent amenability to the laws of functional operation was not taken from it, but occurred to me independently before I had seen the memoir or was acquainted with its contents; and indeed forced itself upon my attention as a means of giving simplicity and generality to my formula for the powers or roots of matrices, published in the *Comptes Rendus* of the Institute for 1882 (Vol. 94, pp. 55, 396). My memoir on Tchebycheff's method concerning the totality of prime numbers within certain limits, was the indirect cause of turning my attention to the subject, as (through the systems of difference-equations therein employed to contract Tchebycheff's limits) I was led to the discovery of the properties of the latent roots of matrices, and had made considerable progress in developing the theory of matrices considered as quantities, when on writing to Prof. Cayley upon the subject he referred me to the memoir in question: all this only proves how far the discovery of the quantitative nature of matrices is removed from being artificial or factitious, but, on the contrary, was bound to be evolved, in the fulness of time, as a necessary sequel to previously acquired cognitions.

Already in Quaternions (which, as will presently be seen, are but the simplest order of matrices viewed under a particular aspect) the example had been given

of Algebra released from the yoke of the commutative principle of multiplication—an emancipation somewhat akin to Lobatchewsky's of Geometry from Euclid's noted empirical axiom; and later on, the Peirces, father and son (but subsequently to 1858) had prefigured the universalization of Hamilton's theory, and had emitted an opinion to the effect that probably all systems of algebraical symbols subject to the associative law of multiplication would be eventually found to be identical with linear transformations of schemata susceptible of matricular representation.

That such must be the case it would be rash to assert; but it is very difficult to conceive how the contrary can be true, or where to seek, outside of the concept of substitution, for matter affording pabulum to the principle of free consociation of successive actions or operations.

Multiplication of Matrices.

A matrix written in the usual form may be regarded as made up of parallels of latitude and of longitude, so that to every term in one matrix corresponds a term of the same latitude and longitude in any other of the same order.

Every matrix possesses a principal axis, viz. the diagonal drawn from the intersection of the first two parallels to the intersection of the last two of latitude and longitude; and by a symmetrical matrix is always to be understood one in which the principal diagonal is the axis of symmetry. If there were ever occasion to consider a symmetrical matrix in which this coincidence does not exist, it might be called improperly symmetrical. This designation might and probably ought to be extended to matrices symmetrical, not merely in regard to the second visible diagonal, but to all the $(\omega - 1)$ rational diagonals of a matrix of the order ω, a rational diagonal being understood to mean any line straight or broken, drawn through ω elements, of which no two have the same latitude or longitude.

The composition of substitutions directly leads to the following rule for the multiplication of matrices. If m, n, be matrices corresponding to substitutions in which m is the antecedent or passive, and n the consequent or active, their product may be denoted by mn (*i. e.* m multiplied by n), and then any term in the product of the two matrices will be equal to its parallel of latitude taken in the antecedent or passive and multiplied by its parallel of longitude taken in the consequent or active matrix. Cauchy has taught us what is to be understood by the product of one rectangular array or matrix by another of the same length and breadth, and we have only to consider the case of rectangles degenerating

each to a single line and column respectively, to understand what is meant by the product of the multiplication of the two parallels spoken of above. It may, however, be sometimes convenient to speak of the *disjunctive product* of two sets of the same number of elements, meaning by this the sum of the products of each element in the one by the corresponding element in the other. Thus $(\lambda l)mn$ denoting the term in mn of latitude λ and longitude l, we have the equation

$$(\lambda l)mn = \lambda m \times ln,$$

where, of course, λm means the λ^{th} parallel of latitude, and ln the l^{th} parallel of longitude in m and n respectively. This notation may be extended so as to express the value of any minor determinant of mn; such minor may obviously be denoted by

$$\begin{array}{l} \lambda_1 l_1,\ \lambda_1 l_2,\ \ldots\ \lambda_1 l_i, \\ \lambda_2 l_1,\ \lambda_2 l_2,\ \ldots\ \lambda_2 l_i, \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \lambda_i l_1,\ \lambda_i l_2,\ \ldots\ \lambda_i l_i, \end{array}$$

and its value will be the product of the two rectangles (in Cauchy's sense) formed respectively by the $\lambda_1, \lambda_2, \ldots \lambda_i$ parallels of latitude in m, and the $l_1, l_2, \ldots l_i$ parallels of longitude n.

Any other definition of multiplication of matrices, such as the rule for multiplying lines by lines, or columns by columns, sins against good method, as being incompatible with the law of consociation, and ought to be inexorably banished from the text-books of the future. It is almost unnecessary to add that by a p^{th} power of a matrix m is to be understood the result of multiplying p m's together; and by the q^{th} root of m, a matrix which multiplied by itself q times produces m: hence we can attach a clear idea to any positive integral or fractional power. The complete extension of the ordinary theory of surds to multinomial quantity will appear a little further on. But it is well at this point to draw attention to the fact that at all events, if M, M' are positive integer powers of the same matrix m, the factors M, M' are convertible, *i. e* $MM' = M'M$, this commutative law being an immediate (too obvious to insist upon) consequence of the associative law of multiplication.

On Zero and Nullity.

The absolute zero for matrices of any order is the matrix all of whose elements are zero. It possesses so far as regards multiplication (and as will presently be evident as regards addition also) the distinguishing property of the

ordinary zero, viz. that when entering into composition with any other matrix, either actively or passively, the product of such composition is itself over again; so that it may be said to absorb into itself any foreign matrix (of its own order) with which it is combined. This is the highest degree of nullity which any matrix can possess, and (regarded as an integer) will be called ω, the order of the matrix. On the other hand, if the matrix has finite content, its nullity will be regarded as zero. Between these two limits the nullity may have any integer value; thus, if its content, *i. e.* its determinant, vanishes without any other special relation existing between its elements, the nullity will be called 1; if all the first minors vanish, 2; and, in general and more precisely, if all the minors of order $\omega - i + 1$ vanish, but the minors of order $\omega - i$ do not all vanish, the nullity will be said to be i: as an example, if the elements are not all zero, but every minor of the second order vanishes, the nullity is $\omega - 1$.

In general, a substitution impressed on a set of variables may be reversed, and the problem of reversal is perfectly determinate; but when the matrix—the schema of the substitution—is affected with any degree of nullity, such reversal becomes indeterminate. Hence the use of the word indeterminate employed by Cayley to characterize matrices affected with any degree of nullity, in which he has been followed by Clifford, who goes a step further in distinguishing the several degrees of indeterminateness from one another.

On Addition and Monomial Multiplication of Matrices.

The sum of two matrices of like order is the matrix of which each element is the sum of the elements of the same latitude and longitude as its own in the component matrices; thus, as stated by anticipation in what precedes, the addition of a zero matrix to any matrix of like order leaves the latter entirely unchanged.

Addition of matrices obviously will be subject to the same two associative and commutative laws as the addition of monomial quantities. This seems to me a sufficient ground for declining to accept *associative* as the distinguishing name of the algebra of multinomial quantity; for the emphasis thereby laid on association would seem to imply the entire absence of the commutative principle from the theory, whereas, although not having a place in multinomial multiplication, it flourishes in full vigor in the not less important, and, so to say, collateral process of multinomial addition. If k is any positive integer, the addition of the same matrix taken k times obviously leads to a matrix of which each

element is k times the corresponding element of the given one; and if p times one matrix is q times another, the elements of the first are obviously $\frac{q}{p}$ into the corresponding ones of the other: hence, if k is any positive monomial quantity, k times a given matrix, by a legitimate use of language, should and will be taken to mean the matrix obtained by multiplying each element in the given one by k. And as the negative of a given matrix ought to mean the matrix which added to the given one should produce the zero-matrix previously defined, the meaning of multiplying a matrix by k may be extended, with the certainty of leading to no contradiction, to the case of any commensurable value of k positive or negative, and consequently, by the usual and valid course of inference, to the case of k being any monomial symbol whatever, whether possessing arithmetical content or not.

On the Multinomial Unit and Scalar Matrix.

On subjecting a matrix of any order ω to a resolution similar to that by which one of the second order may be resolved into a scalar and a vector, it will be shown hereafter that the ω^2 components separate into a group of $\omega^2 - 1$ terms analogous to the vector and to a single term analogous to the scalar of a quaternion. This outstanding single term is of an invariable form, viz. its principal diagonal consists of elements having the same value, which may be called its parameter, and all the other elements are zeros.

A matrix of such form I shall call a scalar. When the parameter is unity it may be termed a multinomial unity and denoted by $\overset{\omega}{\Upsilon}$,* or in place of ω we may write ω dots over Υ, or for greater simplicity when desirable write simply Υ. Any scalar, by virtue of what precedes, is a mere monomial multiplier of some such Υ.

Let $k\Upsilon$ be any scalar of order ω. It will readily be seen, by applying the laws of multiplication and addition previously laid down, that $\phi(k\Upsilon) = \phi(k).\Upsilon$, and that $k\Upsilon.m = m.k\Upsilon = km$.

Thus a scalar possesses all the essential properties of a monomial quantity, and a multinomial unity of ordinary unity; in particular, the faculty of being absorbed in any other coordinate matrix with which it comes in contact. A scalar whose parameter vanishes of course becomes a zero-matrix.

The properties stated of a scalar $k\Upsilon$ serve to show that in all operations into which it enters the Υ may be dropped, and supplied or understood to be

* Perhaps more advantageously by 1_ω. I shall hold myself at liberty in what follows to use whichever of these two notations may appear most convenient in any case as it arises.

supplied at the end of the operations when needed to give homogeneity to an expression. Thus *ex. gr.*

$$(m + h\Upsilon)(m + k\Upsilon) = m^2 + (h + k)\Upsilon m + hk\Upsilon^2 = m^2 + (h + k)\, m + hk\Upsilon\,;$$

but this result may be obtained by the multiplication of $(m + h)(m + k)$, and supplying Υ (or imagining it to be supplied) to the final term in order to preserve the homogeneity of the form. In like manner, 0_ω or 0 with ω points over it may be used to denote the absolute zero of the order ω; but it will be more convenient to use the ordinary 0, having only recourse to the additional notation when thought necessary or desirable in order to make obvious the homogeneity of the terms in any equation or expression. Thus *ex. gr.* such an expression as $m^2 + 2bm + d = 0$, where m is a matrix, say of the 2d order, and b and d monomials, set out in full would read $m^2 + 2bm + d\ddot{\Upsilon} = \ddot{0}$, meaning $m\,.\,m + 2bm + \begin{matrix} d & 0 \\ 0 & d \end{matrix} = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$.

On the Inverse and Negative Powers of a Matrix.

The inverse of a matrix, denoted by m^{-1}, means the matrix which multiplied by m on either side produces multinomial unity. It is a matter of demonstration that when a matrix is non-vacuous (*i. e.* has a finite content or determinant appertaining to it), an inverse to it fulfilling this *double* condition can always be found, and that if the product of mn is unity, so also must be that of nm.

It is a well-known fact, proved in the ordinary theory of determinants, that if every element in the first of two matrices is the logarithmic differential derivative, in respect to its correspondent in the second, of the content of that second, so conversely, every element of the second is the logarithmic derivative, in respect to its correspondent in the first, of the content of the first.

But two such matrices multiplied together in either sense would not give for their product multinomial unity; to obtain this product either matrix must be multiplied indifferently into or by the *transverse* of the other (meaning by the transverse of a matrix, the new matrix obtained by rotating the original one through 180° about its principal diagonal). In other words, if m be a given matrix and n be obtained from it by substituting for each element the logarithmic derivatives of its content in respect to its opposite, then $mn = \overset{\omega}{\Upsilon}$ and $nm = \overset{\omega}{\Upsilon}$, where ω means (as will always be the case throughout these lectures) the order of the matrices concerned. The n which satisfies these two equations (and it cannot satisfy the one without satisfying the other) will be called the inverse of m and be denoted by m^{-1}.

For brevity and suggestiveness it will be advantageous to write in general 1 for $\overset{\omega}{\Upsilon}$ as we write 0 for 0_ω, so that $mn = 1$ will imply $nm = 1 = mn$ and $n = m^{-1}$.

We may define in general (as in monomial algebra) m^{-i} to mean the inverse of m^i, *i. e.* $(m^i)^{-1}$. We shall then have $(m^{-1})^i = m^{-i}$, for $mn.mn = 1$ implies $m.mn.n = mn = 1$ or $m^2n^2 = 1$. Hence $n^2 = m^{-2}$, *i. e.* $(m^{-1})^2 = m^{-2}$. Also since $m^2n^2 = 1$, $m^3n^3 = mn = 1$ or $n^3 = m^{-3}$, *i. e.* $(m^{-1})^3 = m^{-3}$, and so in general for all positive integer values of i, $(m^{-1})^i = m^{-i}$. And, as in monomial algebra, it may now be proved and taken as proved that, for all real values of i and j, whether positive or negative, $m^i.m^j = m^{i+j}$, and the same relation may be assumed to continue when i, j become general quantities. The elements in the inverse to any matrix m all involving the reciprocal of the determinant to m, if D be the content of m we may write $m^{-1} = \frac{1}{D}\mu$, where μ is a matrix all of whose elements are always finite. Hence we come to the important conclusion that for vacuous matrices inverses only exist in idea and are incapable of being realized so as to have an actual existence. In the sequel it will be shown that the inverse is only a single instance of an infinite class of matrices which exist ideally as functions of actual matrices, but are incapable of realization.

Suppose now that M, N are any two matrices such that $MN = 0$ or that $NM = 0$; multiplying each side of the equation by M^{-1} if such expression has an actual existence (*i. e.* if M is non-vacuous), we obtain, from the known properties of zero, $N = 0$, but if M is vacuous no such conclusion can be drawn. So further if $m^i = 0$ (i being any positive integer), it will be seen under the third law of motion that m is necessarily vacuous. Hence from this equation it cannot be inferred that any lower power than the i^{th} of m is necessarily zero.

On the Latent Roots and Different Degrees of Vacuity of Matrices.

If m be any matrix, the augmented matrix $m - \lambda\Upsilon$ or $m - \lambda.1_\omega$ or $m \dot{-} \lambda$ will be found simply by subtracting λ from each element in the principal diagonal of m. The content of this matrix or the same multiplied by -1 or any other constant, I term the latent function to m, which will be an algebraical function of the degree ω in λ (which may be termed the latent variable or carrier); and the ω roots of this function (*i. e.* the ω values of the carrier which annihilate the latent function) I call the latent roots of the unaugmented matrix m. It is obvious from this definition that if λ_1 be any latent root of m, the content of $m - \lambda_1$ will vanish, *i. e.* $m - \lambda_1$ will be vacuous, and conversely that if

$m - \lambda_1$ is vacuous, λ_1 must be one of the latent roots to m. Thus if m is vacuous, one of the latent roots must be zero; if only one of them is zero I call m simply vacuous and say that its vacuity is 1: thus zero vacuity and simple vacuity mean the same thing as zero nullity and simple nullity respectively. More generally if any number i, but not $i+1$, of the latent roots of m are all of them zero, m will be said to have the vacuity i.

By a principal minor determinant to any matrix I mean any minor determinant whose matrix is divided by the principal diagonal into two triangles. It will then easily be seen that if s_i means in general the sum of the principal i^{th} minors to m, and s_0 means the complete determinant, the assertion of m having the vacuity i is exactly coextensive with the assertion that

$$s_0 = 0,\ s_1 = 0,\ s_2 = 0 \ldots s_{i-1} = 0.$$

If the nullity of m is i, *every* q^{th} minor of m when $q < i$ is zero. Hence the vacuity cannot fall short of the nullity, but the converse is not true. A matrix may have any vacuity up to ω inclusive without the nullity being greater than 1. It will hereafter be shown, under the 2d law of motion, that if $\lambda_1, \lambda_2, \ldots \lambda_\omega$ are the ω latent roots of m, then $(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega) = 0$ or say $M = 0$. But it will be interesting even at this early stage to show that a theorem closely approaching this may be deduced from the distinction drawn between vacuous and non-vacuous matrices as regards their possession of real inverses.

I propose to prove instantaneously by this means that at all events $M^{\omega-1} = 0$. It is obvious from any single instance of multiplication that mn and nm are not in general coincident. But if n could be expressed as a linear function of powers of m (including m^0 or 1_ω among such powers), mn and nm must be coincident. If now we take the ω^2 matrices $1, m, m^2, \ldots m^{\omega^2-1}$, n at first blush one would say ought to be expressible as a linear function of these ω^2 quantities determinable by means of the solution of ω^2 linear equations, and can only escape being so expressible in consequence of the fact that these ω^2 powers of m are linearly related. Hence we must have an identical equation of the form

$$Am^{\omega^2-1} + Bm^{\omega^2-2} + Cm^{\omega^2-3} + \ldots + Gm + H = 0_\omega \text{ or say } Fm = 0.$$

If now Fm were supposed to contain any factor other than $m-\lambda_1$, $m-\lambda_2$, $\ldots$ $m-\lambda_\omega$, such factors being non-vacuous may be expelled from Fm; consequently the equation in question must be of the form

$$(m-\lambda_1)^{a_1}(m-\lambda_2)^{a_2}(m-\lambda_\omega)^{a_\omega} = 0,$$

and as the coefficients of the equation in m are necessarily rational we must have $\alpha_1 = \alpha_2, \ldots, \alpha_\omega = \alpha$. Hence $\omega\alpha = \alpha_1 + \alpha_2 + \ldots \alpha_\omega < \omega^2$, and consequently $\alpha < \omega$.

Hence, at all events (since $M^{\omega-1-\theta} = 0$ on multiplication by M^θ gives $M^{\omega-1} = 0$),

$$\{(m-\lambda_1)(m-\lambda_2)\ldots(m-\lambda_\omega)\}^{\omega-1} = M^{\omega-1} = 0. \text{—Q. E. D.}$$

LECTURE II.

On Reduction.

It follows from what has been already shown in Lecture I, when m is a matrix of the second order ($\omega - 1$ being here unity) that $(m-\lambda_1)(m-\lambda_2) = 0$. Understanding by m the matrix $\begin{matrix} t_1, & \tau_1, \\ t_2, & \tau_2, \end{matrix}$ the latent equation to m is

$$\begin{vmatrix} t_1 - \lambda, & \tau_1 \\ t_2 &, \tau_2 - \lambda, \end{vmatrix} = 0,$$

i. e. $$\lambda^2 - (t_1 + \tau_2)\lambda + (t_1\tau_2 - t_2\tau_1) = 0,$$

so that $$m^2 - (t_1 + \tau_2)m + (t_1\tau_2 - t_2\tau_1) = 0,$$

or, using the literation applied to the parametric triangle,

$$m^2 - 2bm + d = 0; \qquad (1)$$

for since the content of $x + ym + zn$ is supposed to be

$$x^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2,$$

that of $-\lambda + m$ will be found by making $z = 0$, $x = -\lambda$, $y = 1$. The variation of equation (1) obtained by taking εn for the increment of m (remembering that the variation of m^2 is $(m + \varepsilon n)(m + \varepsilon n) - m^2$, *i. e.* $\varepsilon(mn + nm)$) gives rise to the identical equation

$$mn + nm - 2bn - 2cm + 2e = 0, \qquad (2)$$

and the variation of this again gives

$$n^2 + n^2 - 2cn - 2cn + 2f = 0,$$

or $n^2 - 2cn + f = 0$, as of course will be obtained immediately from (1) by substituting n, c, f in place of m, b, d.

The parameters c, f, if n represents $\begin{matrix} u_1 v_1 \\ u_2 v_2 \end{matrix}$ are the sum of the principal diagonal elements and the content of u, just as b, d are such sum and content in respect to m.

The parameter e (the connective to d and f) or rather its double $2e$ is obviously the emanant of d in respect to the operator $u_1\delta_{t_1} + u_2\delta_{t_2} + v_1\delta_{\tau_1} + v_2\delta_{\tau_2}$, or, if we please, of f in respect to the inverse operator $t_1\delta_{u_1} + t_2\delta_{u_2} + \tau_1\delta_{v_1} + \tau_2\delta_{v_2}$, *i. e.* is $t_1v_2 + u_1\tau_2 - t_2v_1 - u_2\tau_1$.

With the aid of the *catena* of equations in m, in m and n, and in n, any combination of functions of m and n may be reduced to the standard form

$$Amn + Bm + Cn + D.$$

For, in the first place,

$$\phi m = P(m^2 - 2bm + d) + rm + s = rm + s,$$

and similarly $$\psi n = \rho n + \sigma.$$

Hence the most general combination referred to is expressible as the product of alternating linear functions of m and n, and may therefore be reduced to a sum of terms of which each is a product of alternate powers of m and of n, each of which powers may again be reduced to the form of linear functions, and this process admits of being continually repeated.

Suppose then, at any stage of it, that the greatest number of occurrences of linear functions of m and n in the aggregate of terms is i; then at the next stage of the process the new aggregate will consist of monomial multipliers of one or more *simple* successions of m and n, and of terms in which the number of alternating linear functions never exceeds $i - 1$; hence, eventually we must arrive at a stage when the aggregate will be reduced to a sum of monomial multipliers of simple successions of m and n, every such succession being of the form $$(mn)^q \text{ or } m^{-1}(mn)^q \text{ or } (mn)^q n^{-1} \text{ or } m^{-1}(mn)^q n^{-1}.$$

But
$$\begin{aligned}(mn)^2 &= m.nm.n = -m(mn - 2bn - 2cm + 2e)n \\ &= -m^2n^2 + 2bmn^2 + 2cm^2n - 2emn \\ &= -(2bm - d)(2cn - f) + 2bm(2cn - f) + 2c(2bm - d)n - 2emn \\ &= -(2e - 4bc)mn - df.\end{aligned}$$

Hence $(mn)^2 + 2(e - 2bc)mn + df = 0$.

Hence $(mn)^q = P\{(mn)^2 + 2(e - 2bc)mn + df\} + Amn + B = Amn + B$, where A and B are known functions of $(e - 2bc)$ and f; and therefore

$$m^{-1}(mn)^q = An + Bm^{-1} = An - \frac{B}{d}m + \frac{2Bb}{d}.$$

Similarly $$(mn)^q n^{-1} = Am - \frac{B}{f}n + \frac{2Bc}{f},$$

and $$m^{-1}(mn)^q n^{-1} = A + B(mn)^{-1} = -\frac{B}{df}mn + \left(A - B\frac{2e - 4bc}{df}\right).$$

And this being true (*mut. mut.*) for all values q, it follows that the function expressed by any succession of products of functions of m and n is reducible to the form of a linear expression in m, n, mn, in which the 4 monomial coefficients are known or determinable functions of the parameters to the corpus m, n.

The latent function to any such linear expression, say $Amn+Bm+Cn+D$, may be found in the same way the latent function to mn has been found, viz. as follows:

$$\begin{aligned}(Amn+Bm+Cn+D)^2 &= A^2(mn)^2+AB(mnm+mmn)+AC(mnn+nmn)\\ &\quad+2ADmn+B^2m^2+BC(mn+nm)+C^2n^2+2BDm+2CDn+D^2\\ &= A^2(-2e+4bc)mn-A^2df+ABm(2bn+2cm-2e)\\ &\quad+AC(2bn+2cm-2e)n+2ADmn+B^2m^2+BC(2bn+2cm-2e)+C^2n^2\\ &\quad+2BDm+2CDn+D^2.\end{aligned}$$

Let $(Amn+Bm+Cn+D)^2-2P(Amn+Bm+Cn+D)+Q=0$
be the identical equation to $Amn+Bm+Cn+D$.

The coefficient of mn in the development of the first term being

$$(4bc-2e)A^2+2bAB+2cAC+2AD,$$

and m^2, n^2 being reducible to linear functions of m, n respectively, it follows that

$$P=A(2bc-e)+Bb+Cc+D.$$

To find Q it is only needful to fasten the attention upon the constant terms in the before named development reduced to the standard form. These will be

$$-A^2df-2ABcd-2ACbf-B^2d-2BCe-C^2f+D^2, \text{ say } K,$$

and the constant part in $-2P(Amn+Bm+Cn+D)$ being $-2DP$, it follows that

$$\begin{aligned}Q &= 2AD(2bc-e)+2BDb+2CDc+D^2-K\\ &= A^2df+2ABcd+2ACbf+2AD(2bc-e)\\ &\quad+B^2d+2BCe+C^2f+2BDb+2CDc,\end{aligned}$$

and consequently the latent function $\Lambda^2-2P\Lambda+Q$, of which the algebraical roots are the latent roots of $Amn+Bm+Cn+D$ is completely determined. Thus *ex. gr.*, if the latent function of $m+n$ is required, making $A=D=0$, $B=C=1$, its value will be seen to be $\Lambda^2-2(b+c)\Lambda+d+2e+f=0$, so that the roots will be $b+c\pm\sqrt{(b+c)^2-(d+2e+f)}$.

On Involution.

In general, if m and n be two given binary matrices, and p any third matrix, say

$$m=\begin{matrix}t_1 & t_2\\ t_3 & t_4\end{matrix}, \quad n=\begin{matrix}\tau_1 & \tau_2\\ \tau_3 & \tau_4\end{matrix}, \quad p=\begin{matrix}T_1 & T_2\\ T_3 & T_4\end{matrix},$$

p may be expressed as a linear function of $\ddot{\Upsilon}$, m, n, mn or of $\ddot{\Upsilon}$, m, n, nm. For in order that p may be expressible under the form $A+Bm+Cn+D$, observing that

$$mn=\begin{matrix}t_1\tau_1+t_3\tau_2 & t_2\tau_1+t_4\tau_2\\ t_1\tau_3+t_3\tau_4 & t_2\tau_3+t_4\tau_4\end{matrix},$$

and that $\ddot{\Upsilon}=\begin{matrix}1 & 0\\ 0 & 1\end{matrix}$, it is only necessary to write

$$\begin{aligned} A+Bt_1+C\tau_1+D(t_1\tau_1+t_3\tau_2)&=T_1,\\ Bt_2+C\tau_2+D(t_2\tau_1+t_4\tau_2)&=T_2,\\ Bt_3+C\tau_3+D(t_1\tau_3+t_3\tau_4)&=T_3,\\ D+Bt_4+C\tau_4+D(t_2\tau_3+t_4\tau_4)&=T_4, \end{aligned}$$

and then A, B, C, D may be found by the solution of these four linear equations: and this solution must always be capable of being effected unless the determinant

$$\begin{vmatrix} 1, & t_1, & \tau_1, & t_1\tau_1+t_3\tau_2\\ 0, & t_2, & \tau_2, & t_2\tau_1+t_4\tau_2\\ 0, & t_3, & \tau_3, & t_1\tau_3+t_3\tau_4\\ 1, & t_4, & \tau_4, & t_2\tau_3+t_4\tau_4 \end{vmatrix}$$

vanishes.

When this is the case the matrices m, n, in the order in which they are written, will be said to be in sinistral involution. In like manner, if 1, n, m, nm are linearly related, m, n may be said to be in dextral involution. But it is very easy to see from the identical equation (2) that in this case these two involutions are really identical, for, since $A+Bm+Cn+Dmn=0$, by subtraction

$$A+Bm+Cn-Dnm+2Dcm+2Dbn-2De=0,$$

i. e. $$(A-2eD)+(B+2cD)\,m+(C+2bD)\,n-Dnm=0.$$

The above determinant then will be called the involutant to m, n or n, m, indifferently, for it will be seen, and indeed may be shown, *à priori*, that its value remains absolutely unaltered (not merely to a numerical factor *près*, but in sign and in arithmetical magnitude as well) when the Latin and Greek letters, or which is the same thing, when the matrices m and n are interchanged.

On the Linearform or Summatory Representation of Matrices, and the Multiplication Table to which it gives rise.

This method by which a matrix is robbed as it were of its areal dimensions and represented as a linear sum, first came under my notice incidentally in a communication made some time in the course of the last two years to the Mathematical Society of the Johns Hopkins University, by Mr. C. S. Peirce, who, I presume, had been long familiar with its use. Each element of a matrix in this method is regarded as composed of an ordinary quantity and a symbol denoting its place, just as 1883 may be read $1\theta+8h+8t+3u$, where θ, h, t, u, mean thousands, hundreds, tens, units, or, rather, the places occupied by thousands, hundreds, tens, units, respectively.

Take as an example matrices of the second order, as

$$\begin{matrix} a\,b \\ c\,d \end{matrix} \qquad \begin{matrix} \alpha\,\beta \\ \gamma\,\delta. \end{matrix}$$

These may be denoted respectively by

$$a\lambda + b\mu + c\nu + d\pi, \quad \alpha\lambda + \beta\mu + \gamma\nu + \delta\pi;$$

their product by

$$(a\alpha + c\beta)\lambda + (b\alpha + d\beta)\mu + (a\gamma + c\delta)\nu + (b\gamma + d\delta)\pi,$$

which therefore must be capable of being made identical with

$$\begin{aligned} &a\alpha\lambda^2 + a\beta\lambda\mu + a\gamma\lambda\nu + a\delta\lambda\pi \\ &+ b\alpha\mu\lambda + b\beta\mu^2 + b\gamma\mu\nu + b\delta\mu\pi \\ &+ c\alpha\nu\lambda + c\beta\nu\mu + c\gamma\nu^2 + c\delta\nu\pi \\ &+ d\alpha\pi\lambda + d\beta\pi\mu + d\gamma\pi\nu + d\delta\pi^2, \end{aligned}$$

when a proper system of relations is established between the quadric combinations and the simple powers of λ.

The arguments of like coefficients in the two sums being equated together, there result the equations

$$\begin{aligned} &\lambda^2 = \lambda, \ \lambda\nu = \nu, \ \mu\lambda = \mu, \ \mu\nu = \pi, \\ &\nu\mu = \lambda, \ \nu\pi = \nu, \ \pi\mu = \mu, \ \pi^2 = \pi, \end{aligned}$$

and again, the arguments to the 8 coefficients in the second sum which are not included among the coefficients of the first, being equated to zero, there result the equations

$$\begin{aligned} &\lambda\mu = 0, \ \lambda\pi = 0, \ \mu^2 = 0, \ \mu\pi = 0, \\ &\nu\lambda = 0, \ \nu^2 = 0, \ \pi\lambda = 0, \ \pi\nu = 0. \end{aligned}$$

These 16 equalities may be brought under a single *coup d'œil* by the following multiplication table:

	λ	ν	μ	π
λ	λ	ν	0	0
ν	0	0	λ	ν
μ	μ	π	0	0
π	0	0	μ	π

In like manner it will be found that any matrix of the 3d order, as $\begin{matrix} a & b & c \\ d & e & f, \\ g & h & k \end{matrix}$ regarded as a quantity, may be expressed linearformly by the sum

$$a\lambda + b\mu + c\nu + d\pi + e\rho + f\sigma + g\tau + h\upsilon + k\phi,$$

where the topical symbols are subject to the multiplication table below written:

	λ	π	τ	μ	ρ	υ	ν	σ	ϕ
λ	λ	π	τ	0	0	0	0	0	0
π	0	0	0	λ	π	τ	0	0	0
τ	0	0	0	0	0	0	λ	π	τ
μ	μ	ρ	υ	0	0	0	0	0	0
ρ	0	0	0	μ	ρ	υ	0	0	0
υ	0	0	0	0	0	0	μ	ρ	υ
ν	ν	σ	ϕ	0	0	0	0	0	0
σ	0	0	0	ν	σ	ϕ	0	0	0
ϕ	0	0	0	0	0	0	ν	σ	ϕ

And, in like manner, matrices of any order ω may be expressed linearformly as the sum of ω^2 terms, each consisting of a monomial multiplier of a topical symbol, the entire ω^2 symbols being subject to a multiplication table containing ω^4 places, of which ω^3 will be occupied by the ω^2 simple symbols, each appearing ω times, and the remaining $\omega^4 - \omega^3$ places by the ordinary zero.

This conception applied to quadratic matrices might have served to establish the connection between them and Hamilton's quaternions, regarded as homogeneous functions of $1, i, j, k$, themselves linear functions of the topical symbols λ, μ, ν, π; but the same result may be arrived at somewhat more simply by a method given in a *subsequent* lecture.

On the Corpus *formed by two Independent Matrices of the same order, and the Simple* Parameters *of such Corpus.*

By the latent function of a *corpus* (m, n) we may understand the content or any numerical multiplier of the content of (*i. e.* the determinant to) the matrix $x + ym + zn$, where x, y, z are monomial carriers. This function will be a quantic of the order ω in x, y, z, and in the standard form the coefficient of z^ω may be supposed to be unity, so that it will contain $\dfrac{\omega^2 + 3\omega}{2}$ coefficients, which may be termed the parameters of the corpus.

To fix the ideas, suppose $\omega = 3$ and let the latent function to

$$\begin{matrix} a & b & c & \quad \alpha & \beta & \gamma \\ a' & b' & c' & \quad \alpha' & \beta' & \gamma' \\ a'' & b'' & c'', & \quad \alpha'' & \beta'' & \gamma'' \end{matrix}$$

be called F, where

$$F = x^3 + 3bx^2y + 3cx^2z + 3dxy^2 + 6exyz + 3fxz^2 + gy^3 + 3hy^2z + 3kyz^2 + lz^3.$$

Let m become $m + \varepsilon n$, where ε is a monomial infinitesimal. Then the function to the corpus becomes the content of

$$x + y(m + \varepsilon n) + zn, \; i.\,e.\; x + ym + (z + \varepsilon y)n,$$

and consequently the variation of the function to (m, n) is $\varepsilon y \delta_z F$. If then the *rate* of variation of any of the parameters, when n is the rate of variation of m, be denoted by prefixing to such parameter the symbol E, we shall find

$$Eb = c;\; Ed = 2e;\; Ee = f;\; Eg = 3h;\; Eh = 2k;\; Ek = l;$$

and similarly, if Ǝ, preceding a parameter, be used to indicate its rate of variation corresponding to n's rate of variation being m, then

$$\text{Ǝ}c = b;\; \text{Ǝ}f = 2e;\; \text{Ǝ}e = d;\; \text{Ǝ}l = 3k;\; \text{Ǝ}k = 2h;\; \text{Ǝ}h = g;$$

and the variations of c, f, l, as regards E, and of b, d, g, as regards Ǝ, are of course zero.

By forming the triangle of parameters

$$\begin{matrix} & & & & 1 & & & & \\ & & & b & & c & & & \\ & & d & & e & & f & & \\ & g & & h & & k & & l & \\ p & & q & & r & & s & & t \end{matrix}$$

the law of variations of the parameters of the function to (m, n) (expressed in the ordinary manner by a ternary quantic affected with the proper numerical multipliers) becomes evident, whatever may be the order of the corpus (*i. e.* of the matrices m and n, of which it is constituted): thus *ex. gr.* when $\omega = 4$, in addition to the previous expressions we shall find

$$Ep = 4q,\; Eq = 3r,\; Er = 2s,\; Es = t,\; Et = 0,$$
$$\text{Ǝ}t = 4s,\; \text{Ǝ}s = 3r,\; \text{Ǝ}r = 2q,\; \text{Ǝ}q = p,\; \text{Ǝ}p = 0.$$

By means of the above relations, any identical equation, into which enters one or more matrices, admits of being varied, so as to give rise to an identical equation connecting one additional number of the same.

Scholium.—In what precedes it will have been observed that the matter under consideration has always regard to matrices, or, as we may say, quantities

of a fixed order ω, combined exclusively with one another and with ordinary monomial quantities. Every such combination forms as it were a *clausum* or world of its own, lying completely outside and having no relations with any other. It is, however, possible, and even probable, that as the theory is further evolved, this barrier may be found to give way and the worlds of all the various orders of quantity be brought into relation and intercommunion with one another.

LECTURE III.

On Quantity of the Second Order.

The theory of matrices of the second order seems to me to deserve a special preliminary investigation on various grounds. First, as affording a facile and natural introduction to the general theory (as the study of Conic Sections is usually made to precede that of universal Geometry); secondly, because it presents certain very special features distinguishing it from all other kinds of quantity, such as the coincidence of the two involutants (reminding one of the single image in the case of ordinary refraction as contrasted with the double image seen through iceland spar), or, again, the rational relation between the products of matrices of the second order, in whatever order the factors are introduced in the performance of the multiplication; and thirdly, because the theory of this kind of quantity has already been extensively studied and developed under the name or aspect of Quaternions. Hence it may not be out of place to make the remark that, as it surely would not be logical to seek for the origin of the conception included in the symbol $\sqrt{-1}$ in geometrical considerations, however important its application to geometrical exegesis, so now that an independent algebraical foundation has been discovered for the introduction and use of the symbols employed in Hamilton's theory, it would (it seems to me) be exceedingly illogical and contrary to good method to build the pure theory of the same upon space conceptions; the more so, as it will hereafter be shown that quantities of every order admit of being represented in a mode strictly analogous to that in which quantity of the second order is represented by quaternions, viz., if the order is ω, by ω^2-ions, or as I shall in future say, by *Ions*, of which the geometrical interpretation, although there is little doubt that it exists, is not yet discovered, and it must, it is certain, draw upon the resources of inconceivable space before it can be effected.

(TO BE CONTINUED.)

Note on the Development of an Algebraic Fraction.

By Capt. P. A. MacMahon, R. A.

In the *American Journal of Mathematics*, Vol. V, No. 3, M. Faà de Bruno has considered the development, in ascending powers of x, of the algebraic fraction

$$\phi(x) = \frac{1}{1 + a_1 x + a_2 x^2 + \ldots + a_n x^n},$$

and has obtained the coefficient of x^p in the form of a determinant.

His result may be simply obtained as follows.

For convenience I take the fraction to be

$$f(x) = \frac{1}{1 - a_1 x + a_2 x^2 - \ldots + (-)^n a_n x^n}.$$

Let

$$\begin{aligned} F(y) &= y^n - a_1 y^{n-1} + a_2 y^{n-2} - \ldots + (-)^n a_n \\ &= (y-\alpha)(y-\beta)(y-\gamma)\ldots \end{aligned}$$

so that $\alpha, \beta, \gamma \ldots$ are the roots of the equation $F(y) = 0$;

then

$$\frac{1}{y-\alpha} = \frac{1}{y} + \frac{\alpha}{y^2} + \frac{\alpha^2}{y^3} + \ldots$$

and

$$\frac{1}{F(y)} = \frac{1}{y^n} + \frac{H_1}{y^{n+1}} + \frac{H_2}{y^{n+2}} + \ldots + \frac{H_p}{y^{n+p}} + \ldots$$

wherein H_p represents the sum of the homogeneous symmetric functions, of weight p, of the roots of the equation

$$F(y) = 0.$$

Write now $y = \frac{1}{x}$ and divide both sides of the resulting equation by x^n, thus obtaining

$$\frac{1}{1 - a_1 x + a_2 x^2 - \ldots + (-)^n a_n x^n} = 1 + H_1 x + H_2 x^2 + \ldots + H_p x^p + \ldots$$

It is well known that

$$H_p = \begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_p \\ 1 & a_1 & a_2 & \ldots & a_{p-1} \\ 0 & 1 & a_1 & \ldots & a_{p-2} \\ 0 & 0 & 1 & \ldots & a_{p-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots 1 & a_1 \end{vmatrix}$$

which is equivalent to M. Faà de Bruno's result.

Now $$H_p = \Sigma (-)^{p+\alpha_1+\alpha_2+\alpha_3+\dots} \frac{(\alpha_1+\alpha_2+\alpha_3+\dots)!}{\alpha_1!\,\alpha_2!\,\alpha_3!\dots} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3}\dots$$
the summation extending to all integer, including zero, solutions of the equation $$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = p;$$
consequently we have the result
$$\frac{1}{1 - a_1 x + a_2 x^2 - \dots + (-)^n a_n x^n}$$
$$= \sum_{p=\infty}^{p=0} \sum (-)^{p+k} \frac{k!}{\alpha_1!\,\alpha_2!\,\alpha_3!\dots} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \dots x^p,$$
where $$\alpha_1 + \alpha_2 + \alpha_3 + \dots = k,$$
$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots = p.$$

Royal Military Academy, Woolwich, *October* 2, 1883.

Note by Dr. Franklin.

The general coefficient in the expansion of
$$\frac{1}{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n}$$
is obviously given immediately in the form of a determinant by comparison of coefficients. If the required series is
$$b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n,$$
we have
$$(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)(b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n) = 1,$$
whence
$$\begin{aligned} a_0 b_0 &= 1 \\ a_1 b_0 + a_0 b_1 &= 0 \\ a_2 b_0 + a_1 b_1 + a_0 b_2 &= 0, \text{ etc.,} \end{aligned}$$
whence, solving for the b's,
$$b_p = (-)^p \left(\frac{1}{a_0}\right)^{p+1} \begin{vmatrix} a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_p & a_{p-1} & \dots & \dots & a_0 \end{vmatrix}.$$
The above is so obvious that I have been in the habit of regarding it as the natural method of obtaining the value of H_p, whereas, in the preceding note, Captain MacMahon has reversed the process.

Symmetric Functions of the 13ic.

By Captain P. A. MacMahon, R. A.

The following table represents the symmetric functions of the roots of an equation of the 13th degree, arranged according to Mr. Durfee's plan.

In addition to Professor Cayley's law of symmetry, the following method was carried out in order to ensure the correctness of the numbers: viz. the equation being $x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots + (-)^n a_n = 0$,
and any product of coefficients

$$a_\lambda^l a_\mu^m a_\nu^n \ldots;$$

if the $(r+1)$-agonal weight of the term be defined to be

$$w_{r+1} = l(r+1)_{\lambda-r+1} + m(r+1)_{\mu-r+1} + n(r+1)_{\nu-r+1} + \ldots,$$

wherein $(r+1)_t$ denotes the t^{th} of the $(r+1)$-agonal numbers, then I have elsewhere shown (*vide* Proc. Lond. Math. Soc.) that denoting the sum of all those symmetric functions of weight $w\,(w \not> n)$, which contain r parts in their partitions, by S_w^r,

$$S_w^r = \Sigma\,(-)^{w+k+r-1} \frac{(k-1)!\, w_{r+1}}{l!\, m!\, r!} a_\lambda^l a_\mu^m a_\nu^n \ldots,$$

where

$$w = l\lambda + m\mu + n\nu + \ldots,$$
$$k = l + m + n + \ldots.$$

From this we can, by the law of symmetry, immediately write down the value of any symmetric function of the equation

$$x^n - cx^{n-1} + cx^{n-2} - cx^{n-3} + \ldots + (-)^n c = 0;$$

thus, turning to the partition notation,

$$(\lambda^l.\mu^m.\nu^n\ldots) = \sum_{r=\lambda}^{r=1} (-)^{w_2+k+r-1} \frac{(k-1)!\, w_{r+1}}{l!\, m!\, n!} c^r.$$

w_2, w_{r+1} and k referring to the symmetric function.

In the annexed table, the coefficients were supposed to be each c, and the results compared with those obtained from the above formula.

As a further check, advantage was taken of the fact that every symmetric function (except those whose partitions are composed wholly of units) of the equation

$$x^n - \frac{x^{n-1}}{1!} + \frac{x^{n-2}}{2!} - \frac{x^{n-3}}{3!} + \ldots + (-)^n \frac{1}{n!} = 0,$$

vanishes; the weight of course being $\ngtr n$.

Consequently we must obtain zero, on substituting for each coefficient the reciprocal of the factorial of its suffix.

Royal Military Academy, February 1st, 1884.

	(13)	(12.1)	(11.2)	(10.3)	(9.4)	(8.5)	(7.6)	(11.1^2)	(10.2.1)	(9.3.1)	(9.2^2)	(8.4.1)	(8.3.2)	(7.5.1)	(7.4.2)	(7.3^2)	$(6^2.1)$	(6.5.2)	(6.4.3)	$(5^2.3)$
(13)	+13	−13	−13	−13	−13	−13	−13	+13	+26	+26	+13	+26	+26	+26	+26	+13	+13	+26	+26	+13
(12.1)	−13	+1	+13	+13	+13	+13	+13	−1	−14	−14	−13	−14	−26	−14	−26	−13	−7	−26	−26	−13
(11.2)	−13	+13	−9	+13	+13	+13	+13	−2	−4	−26	+9	−26	−4	−26	−4	−13	−13	−4	−26	−13
(10.3)	−13	+13	+13	−17	+13	+13	+13	−13	+4	+4	−13	−26	+4	−26	−26	+17	−13	−26	+4	+2
(9.4)	−13	+13	+13	+13	−23	+13	+13	−13	−26	+10	+5	+10	−26	−26	+10	−13	−13	−26	+10	−13
(8.5)	−13	+13	+13	+13	+13	−27	+13	−13	−26	−26	−13	+14	+14	+14	−26	−13	−13	+14	−26	+27
(7.6)	−13	+13	+13	+13	+13	+13	−29	−13	−26	−26	−13	−26	−26	+16	+16	+8	+29	+16	+16	−13
(11.1^2)	+13	−1	−2	−13	−13	−13	−13	+1	+3	+14	+2	+14	+15	+14	+15	+13	+7	+15	+26	+13
(10.2.1)	+26	−14	−4	+4	−26	−26	−26	+3	+8	+10	+4	+40	+0	+40	+30	−4	+20	+30	+22	+11
(9.3.1)	+26	−14	−26	+4	+10	−26	−26	+14	+10	+1	+8	+4	+22	+40	+16	−4	+20	+52	−14	+11
(9.2^2)	+13	−13	+9	−13	+5	−13	−13	+2	+4	+8	0	+8	+4	+26	−14	+13	+13	+4	+8	+13
(8.4.1)	+26	−14	−26	−26	+10	+14	−26	+14	+40	+4	+8	−4	+12	0	+16	+26	+20	+12	+16	−14
(8.3.2)	+26	−26	−4	+4	−26	+14	−26	+15	+0	+22	+4	+12	+8	+12	+30	−4	+26	−10	+22	−29
(7.5.1)	+26	−14	−26	−26	−26	+14	+16	+14	+40	+40	+26	0	+12	−7	+10	+5	−22	−30	+10	−14
(7.4.2)	+26	−26	−4	−26	+10	−26	+16	+15	+30	+16	−14	+16	+30	+10	+8	+5	−16	−12	−26	+26
(7.3^2)	+13	−13	−13	+17	−13	−13	+8	+13	−4	−4	+13	+26	−4	+5	+5	+4	−8	+5	−25	−2
$(6^2.1)$	+13	−7	−13	−13	−13	−13	+29	+7	+20	+20	+13	+20	+26	−22	−16	−8	−14	−16	−16	+13
(6.5.2)	+26	−26	−4	−26	−26	+14	+16	+15	+30	+52	+4	+12	−10	−30	−12	+5	−16	+8	+10	−14
(6.4.3)	+26	−26	−26	+4	+10	−26	+16	+26	+22	−14	+8	+16	+22	+10	−26	−25	−16	+10	+16	+11
$(5^2.3)$	+13	−13	−13	+2	−13	+27	−13	+13	+11	+11	+13	−14	−29	−14	+26	−2	+13	−14	+11	+3
(5.4^2)	+13	−13	−13	−13	+23	+7	−13	+13	+26	−10	−5	−30	+6	+6	−10	+13	+13	+6	−10	−7
(10.1^3)	−13	+1	+2	+3	+13	+13	+13	−1	−3	−4	−2	−14	−5	−14	−15	−3	−7	−15	−16	−8
$(9.2.1^2)$	−39	+15	+6	+9	+3	+39	+39	−4	−11	−15	−6	−18	−15	−54	−9	−9	−27	−45	−12	−24
$(8.3.1^2)$	−39	+15	+28	+9	+3	−1	+39	−15	−13	−15	−10	−10	−21	−14	−31	−9	−27	−27	−12	+16
$(8.2^2.1)$	−39	+27	−5	+9	+21	−1	+39	−5	−12	−18	−4	−24	−12	−26	−16	−9	−33	+6	−30	+16
$(7.4.1^2)$	−39	+15	+28	+39	+3	−1	−3	−15	−43	−18	−10	−10	−27	−7	−17	−18	+15	+15	0	+1
(7.3.2.1)	−78	+54	+34	−12	+42	+38	−6	−32	−18	−33	−16	−56	−30	−43	−48	−9	+18	+12	+54	+7
(7.2^3)	−13	+13	−9	+13	−5	+13	−1	−2	−4	−8	0	−8	−4	−12	0	−6	+1	+10	+6	−13
$(6.5.1^2)$	−39	+15	+28	+39	+39	−1	−45	−15	−43	−54	−28	−14	−27	+35	+17	+3	+21	+27	+6	+1
(6.4.2.1)	−78	+54	+34	+48	+6	+38	−48	−32	−78	−30	+2	−52	−42	+34	+30	+15	+24	−6	0	−23
$(6.3^2.1)$	−39	+27	+39	−21	+3	+39	−24	−27	−6	+3	−21	−30	−18	−3	+21	+21	+12	−15	+9	−9
$(6.3.2^2)$	−39	+39	−5	+9	+21	−1	−3	−17	−4	−30	−4	−20	−12	+4	+26	+12	+3	−12	−24	+16
$(5^2.2.1)$	−39	+27	+17	+24	+39	−41	−3	−16	−39	−51	−17	+14	+39	+21	−14	−3	+9	+6	−21	+11

	(5.4^2)	(10.1^3)	$(9.2.1^2)$	$(8.3.1^2)$	$(8.2^2.1)$	$(7.4.1^2)$	$(7.3.2.1)$	(7.2^3)	$(6.5.1^2)$	$(6.4.2.1)$	$(6.3^2.1)$	$(6.3.2^2)$	$(5^2.2.1)$	$(5.4.3.1)$	$(5.4.2^2)$	$(5.3^2.2)$	$(4^3.1)$	(9.1^4)	$(8.2.1^3)$	$(7.3.1^3)$
(13)	+13	−13	−39	−39	−39	−39	−78	−13	−39	−78	−39	−39	−39	−78	−39	−39	−13	+13	+52	+52
(12.1)	−13	+1	+15	+15	+27	+15	+54	+13	+15	+54	+27	+39	+27	+54	+39	+39	+9	−1	−16	−16
(11.2)	−13	+2	+6	+28	−5	+28	+34	−9	+28	+34	+39	−5	+17	+78	−5	+17	+13	−2	−8	−30
(10.3)	−13	+3	+9	+9	+9	+39	−12	+13	+39	+48	−21	+9	+24	+18	+39	−21	+13	−3	−12	−12
(9.4)	+23	+13	+3	+3	+21	+3	+42	−5	+39	+6	+3	+21	+39	−30	−15	+39	−23	−4	−16	−16
(8.5)	+7	+13	+39	−1	−1	−1	+38	+13	−1	+38	+39	−1	−41	−42	−1	−41	−7	−13	−12	−12
(7.6)	−13	+13	+39	+39	+39	−3	−6	−1	−45	−48	−24	−3	−3	+36	−3	+18	+13	−13	−52	−10
(11.1^2)	+13	−1	−4	−15	−5	−15	−32	−2	−15	−32	−27	−17	−16	−54	−17	−28	−9	+1	+5	+16
(10.2.1)	+26	−3	−11	−13	−12	−43	−18	−4	−43	−78	−6	−4	−39	−72	−34	+4	−22	+3	+14	+16
(9.3.1)	−10	−4	−15	−15	−18	−18	−33	−8	−54	−30	+3	−30	−51	+9	−24	−18	+14	+4	+19	+19
(9.2^2)	−5	−2	−6	−10	−4	−10	−16	0	−28	+2	−21	−4	−17	−24	+14	−17	+5	+2	+8	+12
(8.4.1)	−30	−14	−18	−10	−24	−10	−56	−8	−14	−52	−30	−20	+14	+64	+16	+2	+18	+5	+24	+24
(8.3.2)	+6	−5	−15	−21	−12	−27	−30	−4	−27	−42	−18	−12	+39	+24	+6	+36	−6	+5	+20	+26
(7.5.1)	+6	−14	−54	−14	−26	−7	−43	−12	+35	+34	−3	+4	+21	−5	+4	+23	−2	+14	+28	+21
(7.4.2)	−10	−15	−9	−31	−16	−17	−48	0	+17	+30	+21	+26	−14	−6	+6	−35	+10	+6	+24	+32
(7.3^2)	+13	−3	−9	−9	−9	−18	−9	−6	+3	+15	+21	+12	−3	+3	−18	0	−13	+3	+12	+12
$(6^2.1)$	+13	−7	−27	−27	−33	+15	+18	+1	+21	+24	+12	+3	+9	−24	+3	−18	−11	+7	+34	−8
(6.5.2)	+6	−15	−45	−27	+6	+15	+12	+10	+27	−6	−15	−12	+6	+6	−12	+15	−6	+15	+20	0
(6.4.3)	−10	−16	−12	−12	−30	0	+54	+6	+6	0	+9	−24	−21	−18	+18	+3	+10	+7	+28	−14
$(5^2.3)$	−7	−8	−24	+16	+16	+1	+7	−13	+1	−23	−9	+16	+11	−3	+1	−4	+7	+8	−8	−8
(5.4^2)	+7	−13	−3	+17	−1	+17	−22	+5	−19	+14	−3	−1	+1	+10	−5	+1	−7	+4	−4	−4
(10.1^3)	−13	+1	+4	+5	+5	+15	+12	+2	+15	+32	+7	+7	+16	+34	+17	+8	+9	−1	−5	−6
$(9.2.1^2)$	−3	+4	+15	+19	+17	+22	+41	+6	+58	+38	+24	+21	+55	+45	+15	+24	−5	−4	−19	−23
$(8.3.1^2)$	+17	+5	+19	+22	+23	+25	+49	+10	+29	+54	+24	+31	−13	−43	+1	−10	−9	−5	−24	−27
$(8.2^2.1)$	−1	+5	+17	+23	+16	+29	+42	+4	+31	+52	+27	+16	−24	−8	−20	−19	+5	−5	−22	−28
$(7.4.1^2)$	+17	+15	+22	+25	+29	+18	+67	+10	−20	−14	−6	−5	−5	−17	−13	+5	−9	−6	−29	−33
(7.3.2.1)	−22	+12	+41	+49	+42	+67	+88	+16	−9	−46	−42	−38	+2	−10	+24	−1	+14	−12	−53	−61
(7.2^3)	+5	+2	+6	+10	+4	+10	+16	0	0	−16	0	−10	+3	+10	0	+10	−5	−2	−8	−12
$(6.5.1^2)$	−19	+15	+58	+29	+31	−20	−9	0	−32	−26	+3	+1	−17	+17	+1	−1	+11	−15	−33	+5
(6.4.2.1)	+14	+32	+38	+54	+52	−14	−46	−16	−26	+4	−3	+10	+15	+2	−12	+17	−6	−14	−62	+12
$(6.3^2.1)$	−3	+7	+24	+24	+27	−6	−42	0	+3	−3	−24	+12	+12	+18	0	−3	−1	−7	−31	+11
$(6.3.2^2)$	−1	+7	+21	+31	+16	−5	−38	−10	+1	+10	+12	+16	−4	−6	−2	−10	+1	−7	−28	+4
$(5^2.2.1)$	+1	+16	+55	−13	−24	−5	+2	+3	−17	+15	+12	−4	−14	+8	+11	−11	−5	−16	+9	+4

	$(7.2^2.1^2)$	$(6.4.1^3)$	$(6.3.2.1^2)$	$(6.2^3.1)$	$(5^2.1^3)$	$(5.4.2.1^2)$	(8.1^5)	$(7.2.1^4)$	$(6.3.1^4)$	(7.1^6)	$(5.3^2.1^2)$	$(4^2.3.2)$	$(6.2^2.1^3)$	$(6.2.1^5)$	(6.1^7)	$(5.3.2^2.1)$	(5.2^4)	$(5.4.1^4)$	$(5.3.2.1^3)$	$(5.2^3.1^2)$
(13)	+78	+52	+156	+52	+26	+156	-13	-65	-65	+13	+78	-39	-130	+78	-13	+156	+13	-65	-260	-130
(12.1)	-42	-16	-84	-40	-8	-84	+1	+17	+17	-1	-42	+39	+58	-18	+1	-120	-13	+17	+116	+82
(11.2)	-1	-30	-68	+14	-15	-68	+2	+10	+32	-2	-67	+17	+9	-12	+2	-24	+9	+32	+106	-13
(10.3)	-18	-42	-6	-22	-21	-96	+3	+15	+15	-3	+12	+9	+30	-18	+3	-6	-13	+45	+30	+40
(9.4)	-24	-16	-48	-16	-26	-12	+4	+20	+20	-4	-6	-33	+40	-24	+4	-84	+5	+20	+80	+40
(8.5)	-38	-12	-76	-12	+14	+44	+5	+25	+25	-5	+22	+19	+50	-30	+5	+84	-3	-15	-60	-30
(7.6)	-15	+32	+54	+4	+16	+12	+13	+23	-19	-6	-15	-3	-17	+6	0	-30	+1	-19	+8	+11
(11.1^2)	+9	+16	+51	+7	+8	+51	-1	-6	-17	+1	+42	-28	-14	+7	-1	+54	+2	-17	-72	-16
(10.2.1)	+23	+46	+42	+16	+23	+132	-3	-17	-19	+3	+19	-26	-37	+20	-3	+34	+4	-49	-72	-39
(9.3.1)	+33	+22	+63	+26	+29	+63	-4	-23	-23	+4	+12	+24	-52	+27	-4	+81	+8	-26	-101	-59
(9.2^2)	+10	+12	+32	+4	+15	+14	-2	-10	-14	+2	+31	+19	-18	+12	-2	+24	0	-14	-52	-14
(8.4.1)	+42	+24	+84	+32	-6	-40	-5	-29	-29	+5	-20	+14	-66	+34	-5	-20	-2	+11	+28	+6
(8.3.2)	+27	+32	+66	+16	-4	-36	-5	-25	-31	+5	-21	-30	-47	+30	-5	-66	-6	+3	+48	+37
(7.5.1)	+52	-21	-15	-4	-13	-23	-6	-35	+7	+6	-8	-16	+4	-1	0	-20	+2	+12	+21	-3
(7.4.2)	+32	-10	-38	-26	-1	-4	-6	-30	+4	+6	+10	+2	+28	-6	0	+38	0	+4	-18	-6
(7.3^2)	+18	0	-36	-6	0	+12	-3	-15	+6	+3	-12	+12	+12	-3	0	+6	+6	-3	+12	-12
$(6^2.1)$	-3	-14	-18	+2	-7	-12	-7	+1	+7	0	+15	+3	-1	0	0	+12	-1	+7	-8	+1
(6.5.2)	-24	-12	+24	+2	-11	+6	-7	+7	-3	0	-3	+6	-2	0	0	-6	0	+7	-4	+2
(6.4.3)	-21	-2	-12	+18	+5	+12	-7	+7	+7	0	+3	-6	-7	0	0	0	-6	-5	-2	+3
$(5^2.3)$	+8	+7	+1	-3	-4	+1	0	0	0	0	+8	-4	0	0	0	-9	+3	0	0	0
(5.4^2)	+4	-4	+8	-4	+6	-8	0	0	0	0	-4	+3	0	0	0	+4	0	0	0	0
(10.1^3)	-9	-16	-21	-7	-8	-51	+1	+6	+7	-1	-12	+18	+14	-7	+1	-24	-2	+17	+32	+16
$(9.2.1^2)$	-32	-26	-75	-23	-31	-75	+4	+23	+27	-4	-43	-18	+51	-27	+4	-79	-6	+30	+117	+55
$(8.3.1^2)$	-42	-30	-90	-33	+3	+22	+5	+29	+32	-5	+14	-4	+66	-34	+5	+17	0	-5	-19	-5
$(8.2^2.1)$	-33	-34	-82	-20	+2	+22	+5	+27	+33	-5	+10	+11	+55	-32	+5	+46	+6	-1	-28	-27
$(7.4.1^2)$	-51	+9	+12	+3	+5	+24	+6	+35	-3	-6	+6	0	-4	+1	0	-6	0	-8	-5	+3
(7.3.2.1)	-83	+5	+74	+26	+1	-12	+12	+65	-11	-12	+18	+4	-32	+7	0	-7	-6	+2	-15	+12
(7.2^3)	-10	+2	+10	+10	-1	0	+2	+10	0	-2	-10	-5	-10	+2	0	-10	0	0	+10	0
$(6.5.1^2)$	+2	+17	0	-1	+11	+8	+7	-1	-2	0	-7	+2	+1	0	0	+2	0	-7	+3	-1
(6.4.2.1)	+29	+6	+2	-6	+2	+2	+14	-8	-4	0	-6	-2	+3	0	0	-16	+6	-2	+8	-3
$(6.3^2.1)$	+12	+2	+21	-12	-5	-15	+7	-4	-7	0	0	-6	+4	0	0	+3	0	+5	-1	0
$(6.3.2^2)$	+26	-2	-14	-6	+1	+4	+7	-7	+3	0	+2	+5	+2	0	0	+6	0	-1	-2	0
$(5^2.2.1)$	-3	+1	-8	+3	+8	-10	0	0	0	0	-4	-2	0	0	0	+11	-3	0	0	0

	$(5.3.1^5)$	$(5.2^2.1^4)$	$(5.2.1^6)$	(5.1^8)	(4.3^3)	$(4^2.3.1^2)$	$(4^2.2^2.1)$	$(4.3^2.2.1)$	$(4.3.2^3)$	$(4^2.2.1^3)$	$(4.3^2.1^3)$	$(4.3.2^2.1^2)$	$(4.2^4.1)$	$(4^2.1^5)$	$(4.3.2.1^4)$	$(4.2^3.1^3)$	$(4.3.1^6)$	$(4.2^2.1^5)$	$(4.2.1^7)$	(4.1^9)
(13)	+78	+195	−91	+13	−13	+78	+78	+156	+52	−130	−130	−390	−65	+39	+390	+260	−91	−273	+104	−13
(12.1)	−18	− 75	+19	− 1	+13	−42	−60	−120	−52	+ 58	+ 58	+246	+53	−9	−150	−140	+19	+ 93	− 20	+ 1
(11.2)	−34	− 19	+14	− 2	+13	−67	−12	− 90	+14	+ 53	+ 97	+ 93	−23	−17	−148	+ 4	+36	+ 31	− 16	+ 2
(10.3)	−18	− 45	+21	− 3	−17	−33	−48	+ 54	−22	+ 75	0	+ 30	+35	−24	− 60	− 70	+21	+ 63	− 24	+ 3
(9.4)	−24	− 60	+28	− 4	+ 1	+66	+48	+ 24	+20	− 50	− 50	− 60	−25	+ 6	+ 60	+ 40	− 8	− 24	+ 4	0
(8.5)	+10	+ 25	− 5	0	+13	+22	−18	− 36	−12	− 10	− 10	+ 30	+15	+ 5	+ 10	− 20	− 5	+ 5	0	0
(7.6)	+ 6	− 6	0	0	− 8	−36	+ 6	+ 12	+ 4	+ 4	+ 25	− 9	− 5	+ 3	− 12	+ 6	0	0	0	0
(11.1^2)	+18	+ 20	− 8	+ 1	−13	+42	+27	+ 87	+19	− 36	− 58	−114	− 9	+ 9	+ 95	+ 30	−19	− 27	+ 9	− 1
$(10.2.1)$	+22	+ 54	−23	+ 3	+ 4	+64	+62	+ 20	+ 8	− 96	− 35	− 99	−20	+26	+108	+ 76	−25	− 74	+ 26	− 3
$(9.3.1)$	+27	+ 75	−31	+ 4	+16	−42	−18	− 60	+ 2	+ 20	+ 41	+ 39	+ 2	− 3	− 33	− 9	+ 5	+ 6	− 1	0
(9.2^2)	+16	+ 28	−14	+ 2	− 7	− 5	−33	0	−14	+ 19	− 7	+ 42	+14	− 1	− 14	− 28	0	+ 14	− 2	0
$(8.4.1)$	− 6	− 5	+ 1	0	−14	−44	− 8	+ 8	− 8	+ 20	+ 20	+ 14	− 2	− 5	− 22	+ 4	+ 5	− 1	0	0
$(8.3.2)$	− 4	− 28	+ 5	0	+ 4	+12	+24	+ 12	+16	− 8	− 5	− 27	−10	+ 1	+ 8	+ 10	− 1	− 2	0	0
$(7.5.1)$	− 6	+ 1	0	0	− 5	+13	− 6	+ 23	+ 8	+ 8	− 13	− 17	+ 2	− 3	+ 7	− 1	0	0	0	0
$(7.4.2)$	+ 2	+ 2	0	0	+ 7	−13	− 8	+ 10	− 6	+ 11	+ 2	− 12	+ 6	− 3	+ 4	− 2	0	0	0	0
(7.3^2)	− 3	+ 3	0	0	− 4	+12	+ 6	− 12	− 6	− 12	0	+ 12	0	+ 3	− 3	0	0	0	0	0
$(6^2.1)$	0	0	0	0	+ 8	+18	+ 3	− 12	− 4	− 4	− 7	+ 9	− 1	0	0	0	0	0	0	0
$(6.5.2)$	0	0	0	0	− 5	+ 3	+ 6	− 12	+ 2	− 7	+ 3	+ 6	− 2	0	0	0	0	0	0	0
$(6.4.3)$	0	0	0	0	+ 1	− 3	−12	+ 6	+ 6	+ 5	− 1	− 3	0	0	0	0	0	0	0	0
$(5^2.3)$	0	0	0	0	+ 2	− 7	+ 3	+ 6	− 3	0	0	0	0	0	0	0	0	0	0	0
(5.4^2)	0	0	0	0	− 1	+ 4	+ 2	− 4	0	0	0	0	0	0	0	0	0	0	0	0
(10.1^3)	− 8	− 20	+ 8	− 1	+ 3	−27	−27	− 27	− 9	+ 36	+ 18	+ 54	+ 9	− 9	− 45	− 30	+ 9	+ 27	− 9	+ 1
$(9.2.1^2)$	−31	− 74	+31	− 4	− 3	+11	+19	+ 19	+ 9	− 12	− 6	− 30	− 7	+ 1	+ 13	+ 14	− 1	− 7	+ 1	
$(8.3.1^2)$	+ 3	+ 5	− 1	0	− 3	+20	− 1	+ 13	− 5	− 8	− 11	− 5	+ 5	+ 2	+ 10	− 5	− 2	+ 1		
$(8.2^2.1)$	+ 2	+ 18	− 3	0	+ 3	−11	− 5	− 12	− 2	+ 5	+ 6	+ 9	0	− 1	− 6	0	+ 1			
$(7.4.1^2)$	+ 2	− 1	0	0	+ 6	+ 9	+ 9	− 18	+ 3	− 12	+ 3	+ 9	− 3	+ 3	− 3	+ 1				
$(7.3.2.1)$	+ 4	− 3	0	0	− 3	−13	−12	+ 7	+ 2	+ 14	− 1	− 4	0	− 3	+ 1					
(7.2^3)	− 2	0	0	0	0	+ 5	+ 5	0	0	− 5	0	0	0	+ 1						
$(6.5.1^2)$	0	0	0	0	− 3	−13	− 3	+ 7	−1	+ 4	+ 2	− 4	+ 1							
$(6.4.2.1)$	0	0	0	0	− 3	+ 5	+ 2	+ 4	−2	− 1	− 2	+ 1								
$(6.3^2.1)$	0	0	0	0	+ 3	0	+ 6	− 3	0	− 2	+ 1									
$(6.3.2^2)$	0	0	0	0	0	− 1	− 3	0	0	+ 1										
$(5^2.2.1)$	0	0	0	0	+ 3	+ 5	− 1	− 3	+ 1											

	$(3^4.1)$	$(3^3.2^2)$	$(3^3.2.1^2)$	$(3^2.2^3.1)$	(3.2^5)	$(3^3.1^4)$	$(3^2.2^2.1^3)$	$(3.2^4.1^2)$	$(3^2.2.1^5)$	$(3.2^3.1^4)$	$(3^2.1^7)$	$(3.2^2.1^6)$	$(3.2.1^8)$	(3.1^{10})	$(2^6.1)$	$(2^5.1^3)$	$(2^4.1^5)$	$(2^3.1^7)$	$(2^2.1^9)$	$(2.1^{11}.1)$	(1^{13})
(13)	+13	+26	−130	−130	−13	+65	+390	+195	−273	−455	+52	+364	−117	+13	+13	−91	+182	−156	+65	−13	+1
(12.1)	−10	−26	+82	+106	+13	−25	−210	−135	+93	+215	−10	−112	+21	−1	−11	+55	−77	+44	−11	+1	
(11.2)	−13	−4	+75	−2	−9	−43	−104	+36	+97	+15	−19	−45	+18	−2	+9	−30	+27	−9	+1		
(10.3)	+17	+19	−80	−35	+7	+20	+105	0	−42	−35	+3	+21	−3	0	−7	+14	−7	+1			
(9.4)	−1	−20	+10	+40	−5	+10	−30	−15	−6	+20	+2	−4	0	0	+5	−5	+1				
(8.5)	−13	+14	+30	−30	+3	−5	−10	+15	+5	−5	0	0	0	0	−3	+1					
(7.6)	+8	−5	−17	+11	−1	−2	+9	−6	0	0	0	0	0	0	+1						
(11.1^2)	+10	+15	−60	−40	−2	+25	+100	+25	−60	−50	+10	+35	−10	+1							
(10.2.1)	−7	−15	+33	+31	+2	−7	−46	−16	+15	+20	−1	−8	+1								
(9.3.1)	−10	+1	+28	−8	−2	−13	−12	+9	+12	−6	−2	+1									
(9.2^2)	+7	+7	−21	−7	0	+7	+14	0	−7	0	+1										
(8.4.1)	+11	+6	−24	−2	+2	+3	+12	−4	−3	+1											
(8.3.2)	−4	−7	+9	+5	0	−1	−5	0	+1												
(7.5.1)	+2	−9	0	+8	−2	+2	−4	+1													
(7.4.2)	−7	+3	+6	−3	0	−2	+1														
(7.3^2)	+4	+2	−4	0	0	+1															
$(6^2.1)$	−5	+5	+5	−5	+1																
(6.5.2)	+5	−1	−3	+1																	
(6.4.3)	−1	−2	+1																		
$(5^2.3)$	−2	+1																			
(5.4^2)	+1																				

	(13)	(12.1)	(11.2)	(10.3)	(9.4)	(8.5)	(7.6)	(11.1^2)	(10.2.1)	(9.3.1)	(9.2^2)	(8.4.1)	(8.3.2)	(7.5.1)	(7.4.2)	(7.3^2)	$(6^2.1)$	(6.5.2)	(6.4.3)	$(5^2.3)$
(5.4.3.1)	−78	+54	+78	+18	−30	−42	+36	−54	−72	+9	−24	+64	+24	−5	−6	+3	−24	+6	−18	−3
$(5.4.2^2)$	−39	+39	−5	+39	−15	−1	−3	−17	−34	−24	+14	+16	+6	+4	+6	−18	+3	−12	+18	+1
$(5.3^2.2)$	−39	+39	+17	−21	+39	−41	+18	−28	+4	−18	−17	+2	+36	+23	−35	0	−18	+15	+3	−4
$(4^3.1)$	−13	+9	+13	+13	−23	−7	+13	−9	−22	+14	+5	+18	−6	−2	+10	−13	−11	−6	+10	+7
(9.1^4)	+13	−1	−2	−3	−4	−13	−13	+1	+3	+4	+2	+5	+5	+14	+6	+3	+7	+15	+7	+8
$(8.2.1^3)$	+52	−16	−8	−12	−16	−12	−52	+5	+14	+19	+8	+24	+20	+28	+24	+12	+34	+20	+28	−8
$(7.3.1^3)$	+52	−16	−30	−12	−16	−12	−10	+16	+16	+19	+12	+24	+26	+21	+32	+12	−8	0	−14	−8
$(7.2^2.1^2)$	+78	−42	−1	−18	−24	−38	−15	+9	+23	+33	+10	+42	+27	+52	+32	+18	−3	−24	−21	+8
$(6.4.1^3)$	+52	−16	−30	−42	−16	−12	+32	+16	+46	+22	+12	+24	+32	−21	−10	0	−14	−12	−2	+7
$(6.3.2.1^2)$	+156	−84	−68	−6	−48	−76	+54	+51	+42	+63	+32	+84	+66	−15	−38	−36	−18	+24	−12	+1
$(6.2^3.1)$	+52	−40	+14	−22	−16	−12	+4	+7	+16	+26	+4	+32	+16	−4	−26	−6	+2	+2	+18	−3
$(5^2.1^3)$	+26	−8	−15	−21	−26	+14	+16	+8	+23	+29	+15	−6	−4	−13	−1	0	−7	−11	+5	−4
$(5.4.2.1^2)$	+156	−84	−68	−96	−12	+44	+12	+51	+132	+63	+14	−40	−36	−23	−4	+12	−12	+6	+12	+1
(8.1^5)	−13	+1	+2	+3	+4	+5	+13	−1	−3	−4	−2	−5	−5	−6	−6	−3	−7	−7	−7	0
$(7.2.1^4)$	−65	+17	+10	+15	+20	+25	+23	−6	−17	−23	−10	−29	−25	−35	−30	−15	+1	+7	+7	0
$(6.3.1^4)$	−65	+17	+32	+15	+20	+25	−19	−17	−19	−23	−14	−29	−31	+7	+4	+6	+7	−3	+7	0
(7.1^6)	+13	−1	−2	−3	−4	−5	−6	+1	+3	+4	+2	+5	+5	+6	+6	+3	0	0	0	0
$(5.3^2.1^2)$	+78	−42	−67	+12	−6	+22	−15	+42	+19	+12	+31	−20	−21	−8	+10	−12	+15	−3	+3	+8
$(4^2.3.2)$	−39	+39	+17	+9	−33	+19	−3	−28	−26	+24	+19	+14	−30	−16	+2	+12	+3	+6	−6	−4
$(6.2^2.1^3)$	−130	+58	+9	+30	+40	+50	−17	−14	−37	−52	−18	−66	−47	+4	+28	+12	−1	−2	−7	0
$(6.2.1^5)$	+78	−18	−12	−18	−24	−30	+6	+7	+20	+27	+12	+34	+30	−1	−6	−3	0	0	0	0
(6.1^7)	−13	+1	+2	+3	+4	+5	0	−1	−3	−4	−2	−5	−5	0	0	0	0	0	0	0
$(5.3.2^2.1)$	+156	−120	−24	−6	−84	+84	−30	+54	+34	+81	+24	−20	−66	−20	+38	+6	+12	−6	0	−9
(5.2^4)	+13	−13	+9	−13	+5	−3	+1	+2	+4	+8	0	−2	−6	+2	0	+6	−1	0	−6	+3
$(5.4.1^4)$	−65	+17	+32	+45	+20	−15	−19	−17	−49	−26	−14	+11	+3	+12	+4	−3	+7	+7	−5	0
$(5.3.2.1^3)$	−260	+116	+106	+30	+80	−60	+8	−72	−72	−101	−52	+28	+48	+21	−18	+12	−8	−4	−2	0
$(5.2^3.1^2)$	−130	+82	−13	+40	+40	−30	+11	−16	−39	−59	−14	+6	+37	−3	−6	−12	+1	+2	+3	0
$(5.3.1^5)$	+78	−18	−34	−18	−24	+10	+6	+18	+22	+27	+16	−6	−4	−6	+2	−3	0	0	0	0
$(5.2^2.1^4)$	+195	−75	−19	−45	−60	+25	−6	+20	+54	+75	+28	−5	−28	+1	+2	+3	0	0	0	0
$(5.2.1^6)$	−91	+19	+14	+21	+28	−5	0	−8	−23	−31	−14	+1	+5	0	0	0	0	0	0	0
(5.1^8)	+13	−1	−2	−3	−4	0	0	+1	+3	+4	+2	0	0	0	0	0	0	0	0	0
(4.3^3)	−13	+13	+13	−17	+1	+13	−8	−13	+4	+16	−7	−14	+4	−5	+7	−4	+8	−5	+1	+2
$(4^2.3.1^2)$	+78	−42	−67	−33	+66	+22	−36	+42	+64	−42	−5	−44	+12	+13	−13	+12	+18	+3	−3	−7
$(4^2.2^2.1)$	+78	−60	−12	−48	+48	−18	+6	+27	+62	−18	−33	−8	+24	−6	−8	+6	+3	+6	−12	+3

	(5.4²)	(10.1³)	(9.2.1²)	(8.3.1²)	(8.2².1)	(7.4.1²)	(7.3.2.1)	(7.2³)	(6.5.1²)	(6.4.2.1)	(6.3².1)	(6.3.2²)	(5².2.1)	(5.4.3.1)	(5.4.2²)	(5.3².2)	(4³.1)	(9.1⁴)	(8.2.1³)	(7.3.1³)
(5.4.3.1)	+10	+34	+45	−43	−8	−17	−10	+10	+17	+2	+18	−6	+8	−10	−10	+3	−2	−16	+9	+16
(5.4.2²)	−5	+17	+15	+1	−20	−13	+24	0	+1	−12	0	−2	+11	−10	0	+2	+5	−8	+8	−4
(5.3².2)	+1	+8	+24	−10	−19	+5	−1	+10	−1	+17	−3	−10	−11	+3	+2	+4	−1	−8	+8	+2
(4³.1)	−7	+9	−5	−9	+5	−9	+14	−5	+11	−6	−1	+1	−5	−2	+5	−1	+3	0	0	0
(9.1⁴)	+4	−1	−4	−5	−5	−6	−12	−2	−15	−14	−7	−7	−16	−16	−8	−8	0	+1	+5	+6
(8.2.1³)	−4	−5	−19	−24	−22	−29	−53	−8	−33	−62	−31	−28	+9	+9	+8	+8	0	+5	+24	+29
(7.3.1³)	−4	−6	−23	−27	−28	−33	−61	−12	+5	+12	+11	+4	+4	+16	−4	+2	0	+6	+29	+33
(7.2².1²)	+4	−9	−32	−42	−33	−51	−83	−10	+2	+29	+12	+26	−3	−9	−2	−5	0	+9	+41	+51
(6.4.1³)	−4	−16	−26	−30	−34	+9	+5	+2	+17	+6	+2	−2	+1	+1	+8	−7	0	+7	+34	−3
(6.3.2.1²)	+8	−21	−75	−90	−82	+12	+74	+10	0	+2	+21	−14	−8	−10	−4	+5	0	+21	+96	−15
(6.2³.1)	−4	−7	−23	−33	−20	+3	+26	+10	−1	−6	−12	−6	+3	+4	+2	0	0	+7	+30	−2
(5².1³)	+6	−8	−31	+3	+2	+5	+1	−1	+11	+2	−5	+1	+8	−8	−4	+4	0	+8	−1	−2
(5.4.2.1²)	−8	−51	−75	+22	+22	+24	−12	0	+8	+2	−15	+4	−10	+10	0	−1	0	+24	−10	−6
(8.1⁵)	0	+1	+4	+5	+5	+6	+12	+2	+7	+14	+7	+7	0	0	0	0	0	−1	−5	−6
(7.2.1⁴)	0	+6	+23	+29	+27	+35	+65	+10	−1	−8	−4	−7	0	0	0	0	0	−6	−29	−35
(6.3.1⁴)	0	+7	+27	+32	+33	−3	−11	0	−2	−4	−7	+3	0	0	0	0	0	−7	−34	+3
(7.1⁶)	0	−1	−4	−5	−5	−6	−12	−2	0	0	0	0	0	0	0	0	0	+1	+5	+6
(5.3².1²)	−4	−12	−43	+14	+10	+6	+18	−10	−7	−6	0	+2	−4	0	+4	−2	0	+12	−5	−9
(4².3.2)	+3	+18	−18	−4	+11	0	+4	−5	+2	−2	−6	+5	−2	+8	−1	−2	−3	0	0	0
(6.2².1³)	0	+14	+51	+66	+55	−4	−32	−10	+1	+3	+4	+2	0	0	0	0	0	−14	−65	+4
(6.2.1⁵)	0	−7	−27	−34	−32	+1	+7	+2	0	0	0	0	0	0	0	0	0	+7	+34	−1
(6.1⁷)	0	−1	+4	+5	+5	0	0	0	0	0	0	0	0	0	0	0	0	−1	−5	0
(5.3.2².1)	+4	−24	−79	+17	+46	−6	−7	−10	+2	−16	+3	+6	+11	−1	−2	0	0	+24	−17	0
(5.2⁴)	0	−2	−6	0	+6	0	−6	0	0	+6	0	0	−3	0	0	0	0	+2	−2	+2
(5.4.1⁴)	0	+17	+30	−5	−1	−8	+2	0	−7	−2	+5	−1	0	0	0	0	0	−8	+1	+2
(5.3.2.1³)	0	+32	+117	−19	−28	−5	−15	+10	+3	+8	−1	−2	0	0	0	0	0	−32	+11	+8
(5.2³.1²)	0	+16	+55	−5	−27	+3	+12	0	−1	−3	0	0	0	0	0	0	0	−16	+9	−4
(5.3.1⁵)	0	−8	−31	+3	+2	+2	+4	−2	0	0	0	0	0	0	0	0	0	+8	−1	−2
(5.2².1⁴)	0	−20	−74	+5	+18	−1	−3	0	0	0	0	0	0	0	0	0	0	+20	−6	+1
(5.2.1⁶)	0	+8	+31	−1	−3	0	0	0	0	0	0	0	0	0	0	0	0	−8	+1	
(5.1⁸)	0	−1	−4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	+1		
(4.3³)	−1	+3	−3	−3	+3	+6	−3	0	−3	−3	+3	0	+3	−3	0	0	+1			
(4².3.1²)	+4	−27	+11	+20	−11	+9	−13	+5	−13	+5	0	−1	+5	−1	−2	+1				
(4².2².1)	+2	−27	+19	−1	−5	+9	−12	+5	−3	+2	+6	−3	−1	−2	+1					

	$(7.2^2.1^2)$	$(6.4.1^3)$	$(6.3.2.1^2)$	$(6.2^3.1)$	$(5^2.1^3)$	$(5.4.2.1^2)$	(8.1^5)	$(7.2.1^4)$	$(6.3.1^4)$	(7.1^6)	$(5.3^2.1^2)$	$(4^2.3.2)$	$(6.2^2.1^3)$	$(6.2.1^5)$	(6.1^7)	$(5.3.2^2.1)$	(5.2^4)	$(5.4.1^4)$	$(5.3.2.1^3)$	$(5.2^3.1^2)$
$(5.4.3.1)$	−9	+1	−10	+4	−8	+10	0	0	0	0	0	+8	0	0	0	−1	0	0	0	0
$(5.4.2^2)$	−2	+8	−4	+2	−4	0	0	0	0	0	+4	−1	0	0	0	−2	0	0	0	0
$(5.3^2.2)$	−5	−7	+5	0	+4	−1	0	0	0	0	−2	−2	0	0	0	0	0	0	0	0
$(4^3.1)$	0	0	0	0	0	0	0	0	0	0	0	−3	0	0	0	0	0	0	0	0
(9.1^4)	+9	+7	+21	+7	+8	+24	−1	−6	−7	+1	+12	0	−14	+7	−1	+24	+2	−8	−32	−16
$(8.2.1^3)$	+41	+34	+96	+30	−1	−10	−5	−29	−34	+5	−5	0	−65	+34	−5	−17	−2	+1	+11	+9
$(7.3.1^3)$	+51	−3	−15	−2	−2	−6	−6	−35	+3	+6	−9	0	+4	−1	0	0	+2	+2	+8	−4
$(7.2^2.1^2)$	+65	−3	−32	−20	+1	+4	−9	−50	+3	+9	+5	0	+20	−4	0	+5	0	−1	−5	0
$(6.4.1^3)$	−3	−5	−3	+2	−3	−9	−7	+1	+2	0	+3	0	−1	0	0	+6	−2	+3	−3	+1
$(6.3.2.1^2)$	−32	−3	−17	+12	+3	+11	−21	+9	+6	0	−1	0	−4	0	0	−3	0	−3	+1	
$(6.2^3.1)$	−20	+2	+12	0	−1	−4	−7	+5	−3	0	0	0	0	0	0	0	0	+1		
$(5^2.1^3)$	+1	−3	+3	−1	−4	+4	0	0	0	0	+2	0	0	0	0	−4	+1			
$(5.4.2.1^2)$	+4	−9	+11	−4	+4	−1	0	0	0	0	−2	0	0	0	0	+1				
(8.1^5)	−9	−7	−21	−7	0	0	+1	+6	+7	−1	0	0	+14	−7	+1					
$(7.2.1^4)$	−50	+1	+9	+5	0	0	+6	+35	−1	−6	0	0	−5	+1						
$(6.3.1^4)$	+3	+2	+6	−3	0	0	+7	−1	−2	0	0	0	+1							
(7.1^6)	+9	0	0	0	0	0	−1	−6	0	+1										
$(5.3^2.1^2)$	+5	+3	−1	0	+2	−2	0	0	0	0	+1									
$(4^2.3.2)$	0	0	0	0	0	0	0	0	0	0	0	+1								
$(6.2^2.1^3)$	+20	−1	−4	0	0	0	+14	−5	+1											
$(6.2.1^5)$	−4	0	0	0	0	0	−7	+1												
(6.1^7)	0	0	0	0	0	0	+1													
$(5.3.2^2.1)$	+5	+6	−3	0	−4	+1														
(5.2^4)	0	−2	0	0	+1															
$(5.4.1^4)$	−1	+3	−3	+1																
$(5.3.2.1^3)$	−5	−3	+1																	
$(5.2^3.1^2)$	0	+1																		
$(5.3.1^5)$	+1																			

	$(5.3.1^5)$	$(5.2^2.1^4)$	$(5.2.1^6)$	(5.1^8)	(4.3^3)	$(4^2.3.1^2)$	$(4^2.2^2.1)$	$(4.3^2.2.1)$
$(5.4.3.1)$	0	0	0	0	−3	−1	−2	+1
$(5.4.2^2)$	0	0	0	0	0	−2	+1	
$(5.3^2.2)$	0	0	0	0	0	+1		
$(4^3.1)$	0	0	0	0	+1			
(9.1^4)	+8	+20	−8	+1				
$(8.2.1^3)$	−1	−6	+1					
$(7.3.1^3)$	−2	+1						
$(7.2^2.1^2)$	+1							

	(13)	(12.1)	(11.2)	$(10.3.1)$	(9.4)	(8.5)	(7.6)	(11.1^2)	$(10.2.1)$	$(9.3.1)$	(9.2^2)	$(8.4.1)$	$(8.3.2)$	$(7.5.1)$	$(7.4.2)$	(7.3^2)	$(6^2.1)$	$(6.5.2)$	$(6.4.3)$	$(5^2.3)$
$(4.3^2.2.1)$	+156	−120	− 90	+ 54	+24	−36	+12	+ 87	+ 20	−60	0	+ 8	+12	+23	+10	−12	−12	−12	+ 6	+ 6
$(4.3.2^3)$	+ 52	− 52	+ 14	− 22	+20	−12	+ 4	+ 19	+ 8	+ 2	−14	− 8	+16	+ 8	− 6	− 6	− 4	+ 2	+ 6	− 3
$(4^2.2.1^3)$	−130	+ 58	+ 53	+ 75	−50	−10	+ 4	− 36	− 96	+20	+19	+20	− 8	+ 8	+11	−12	− 4	− 7	+ 5	0
$(4.3^2.1^3)$	−130	+ 58	+ 97	0	−50	−10	+25	− 58	− 35	+41	− 7	+20	− 5	−13	+ 2	0	− 7	+ 3	− 1	0
$(4.3.2^2.1^2)$	−390	+246	+ 93	+ 30	−60	+30	− 9	−114	− 99	+39	+42	+14	−27	−17	−12	+12	+ 9	+ 6	− 3	0
$(4.2^4.1)$	− 65	+ 53	− 23	+ 35	−25	+15	− 5	− 9	− 20	+ 2	+14	− 2	−10	+ 2	+ 6	0	− 1	− 2	0	0
$(4^2.1^5)$	+ 39	− 9	− 17	− 24	+ 6	+ 5	+ 3	+ 9	+ 26	− 3	− 1	− 5	+ 1	− 3	− 3	+ 3	0	0	0	0
$(4.3.2.1^4)$	+390	−150	−148	− 60	+60	+10	−12	+ 95	+108	−33	−14	−22	+ 8	+ 7	+ 4	− 3	0	0	0	0
$(4.2^3.1^3)$	+260	−140	+ 4	− 70	+40	−20	+ 6	+ 30	+ 76	− 9	−28	+ 4	+10	− 1	− 2	0	0	0	0	0
$(4.3.1^6)$	− 91	+ 19	+ 36	+ 21	− 8	− 5	0	− 19	− 25	+ 5	0	+ 5	− 1	0	0	0	0	0	0	0
$(4.2^2.1^5)$	−273	+ 93	+ 31	+ 63	−24	+ 5	0	− 27	− 74	+ 6	+14	− 1	− 2	0	0	0	0	0	0	0
$(4.2.1^7)$	+104	− 20	− 16	− 24	+ 4	0	0	+ 9	+ 26	− 1	− 2	0	0	0	0	0	0	0	0	0
(4.1^9)	− 13	+ 1	+ 2	+ 3	0	0	0	− 1	− 3	0	0	0	0	0	0	0	0	0	0	0
$(3^4.1)$	+ 13	− 10	− 13	+ 17	− 1	−13	+ 8	+ 10	− 7	−10	+ 7	+11	− 4	+ 2	− 7	+ 4	− 5	+ 5	− 1	− 2
$(3^3.2^2)$	+ 26	− 26	− 4	+ 19	−20	+14	− 5	+ 15	− 15	+ 1	+ 7	+ 6	− 7	− 9	+ 3	+ 2	+ 5	− 1	− 2	+ 1
$(3^3.2.1^2)$	−130	+ 82	+ 75	− 80	+10	+30	−17	− 60	+ 33	+28	−21	−24	+ 9	0	+ 6	− 4	+ 5	− 3	+ 1	
$(3^2.2^3.1)$	−130	+106	− 2	− 35	+40	−30	+11	− 40	+ 31	− 8	− 7	− 2	+ 5	+ 8	− 3	0	− 5	+ 1		
(3.2^5)	− 13	+ 13	− 9	+ 7	− 5	+ 3	− 1	− 2	+ 2	− 2	0	+ 2	0	− 2	0	0	+ 1			
$(3^3.1^4)$	+ 65	− 25	− 43	+ 20	+10	− 5	− 2	+ 25	− 7	−13	+ 7	+ 3	− 1	+ 2	− 2	+ 1				
$(3^2.2^2.1^3)$	+390	−210	−104	+105	−30	−10	+ 9	+100	− 46	−12	+14	+12	− 5	− 4	+ 1					
$(3.2^4.1^2)$	+195	−135	+ 36	0	−15	+15	− 6	+ 25	− 16	+ 9	0	− 4	0	+ 1						
$(3^2.2.1^5)$	−273	+ 93	+ 97	− 42	− 6	+ 5	0	− 60	+ 15	+12	− 7	− 3	+ 1							
$(3.2^3.1^4)$	−455	+215	+ 15	− 35	+20	− 5	0	− 50	+ 20	− 6	0	+ 1								
$(3^2.1^7)$	+ 52	− 10	− 19	+ 3	+ 2	0	0	+ 10	− 1	− 2	+ 1									
$(3.2^2.1^6)$	+364	−112	− 45	+ 21	− 4	0	0	+ 35	− 8	+ 1										
$(3.2.1^8)$	−117	+ 21	+ 18	− 3	0	0	0	− 10	+ 1											
(3.1^{10})	+ 13	− 1	− 2	0	0	0	0	+ 1												
$(2^6.1)$	+ 13	− 11	+ 9	− 7	+ 5	− 3	+ 1													
$(2^5.1^3)$	− 91	+ 55	− 30	+ 14	− 5	+ 1														
$(2^4.1^5)$	+182	− 77	+ 27	− 7	+ 1															
$(2^3.1^7)$	−156	+ 44	− 9	+ 1																
$(2^2.1^9)$	+ 65	− 11	+ 1																	
(2.1^{11})	− 13	+ 1																		
(1^{13})	+ 1																			

	(5.4^2)	(10.1^3)	$(9.2.1^2)$	$(8.3.1^2)$	$(8.2^2.1)$	$(7.4.1^2)$	$(7.3.2.1)$	(7.2^3)	$(6.5.1^2)$	$(6.4.2.1)$	$(6.3^2.1)$	$(6.3.2^2)$	$(5^2.2.1)$	
$(4.3^2.2.1)$	-4	-27	$+19$	$+13$	-12	-18	$+7$	0	$+7$	$+4$	-3	0	-3	$+1$
$(4.3.2^3)$	0	-9	$+9$	-5	-2	$+3$	$+2$	0	-1	-2	0	0	$+1$	
$(4^2.2.1^3)$	0	$+36$	-12	-8	$+5$	-12	$+14$	-5	$+4$	-1	-2	$+1$		
$(4.3^2.1^3)$	0	$+18$	-6	-11	$+6$	$+3$	-1	0	$+2$	-2	$+1$			
$(4.3.2^2.1^2)$	0	$+54$	-30	-5	$+9$	$+9$	-4	0	-4	$+1$				
$(4.2^4.1)$	0	$+9$	-7	$+5$	0	-3	0	0	$+1$					
$(4^2.1^5)$	0	-9	$+1$	$+2$	-1	$+3$	-3	$+1$						
$(4.3.2.1^4)$	0	-45	$+13$	$+10$	-6	-3	$+1$							
$(4.2^3.1^3)$	0	-30	$+14$	-5	0	$+1$								
$(4.3.1^6)$	0	$+9$	-1	-2	$+1$									
$(4.2^2.1^5)$	0	$+27$	-7	$+1$										
$(4.2.1^7)$	0	-9	$+1$											
(4.1^9)	0	$+1$												
$(3^4.1)$	$+1$													

On the Resolution of Equations of the Fifth Degree.

By Emory McClintock.

Two papers published by Mr. George Paxton Young in the *American Journal of Mathematics* (vol. VI, pp. 65–114) contain discoveries of high importance in the theory of equations. It is a necessary consequence of such publications that others should be attracted to the same field, and I will say at once that what I have now to present is merely in continuation of the work done by Mr. Young.

To relieve the reader from referring repeatedly to Mr. Young's paper on the resolution of equations of the fifth degree, I will first give a brief summary of its main features. The roots (r_1, r_2, r_3, r_4, r_5) of the general equation $x^5+p_2x^3+p_3x^2+p_4x+p_5=0$ being, in accordance with Lagrange's theory, of the form $r_1=u_1+u_2+u_3+u_4$, and so on, where u_1^5, u_2^5, u_3^5, u_4^5, are given fractions of the roots of a biquadratic equation, Mr. Young shows, first, that the latter have, in the most general case, this form,

$$\left.\begin{array}{ll} u_1^5=\mathrm{B}+\mathrm{B}'\sqrt{z}+\sqrt{s}, & u_2^5=\mathrm{B}-\mathrm{B}'\sqrt{z}+\sqrt{s_1}, \\ u_4^5=\mathrm{B}+\mathrm{B}'\sqrt{z}-\sqrt{s}, & u_3^5=\mathrm{B}-\mathrm{B}'\sqrt{z}-\sqrt{s_1}, \end{array}\right\} \tag{1}$$

where $z=1+e^2$, $s=hz+h\sqrt{z}$, $s_1=hz-h\sqrt{z}$, and e, h, B, and B′ are rational; that

$$u_1u_4=g+a\sqrt{z}, \qquad u_2u_3=g-a\sqrt{z}, \tag{2}$$

where $g=-\frac{1}{10}p_2$, and a is rational; and that

$$\left.\begin{array}{l} u_1^2u_3=k+c\sqrt{z}+(\theta+\phi\sqrt{z})\sqrt{s}, \\ u_4^2u_2=k+c\sqrt{z}-(\theta+\phi\sqrt{z})\sqrt{s}, \\ u_2^2u_1=k-c\sqrt{z}+(\theta-\phi\sqrt{z})\sqrt{s_1}, \\ u_3^2u_4=k-c\sqrt{z}-(\theta-\phi\sqrt{z})\sqrt{s_1}, \end{array}\right\} \tag{3}$$

where $k=-\frac{1}{20}p_3$, and c, θ, and ϕ are rational. From these and others he derives the six equations following:

$$p_4(g^2-a^2z)=(g^2-a^2z)(5g^2+15a^2z)-20g\,(k^2-c^2z)-20ahcz\,(\theta^2-\phi^2z),$$
$$p^5=-4\text{B}+40acz,$$
$$\text{B}''=1,$$
$$\text{B}'''=0,$$
$$hz(\theta^2+\phi^2z+2\theta\phi)=k^2+c^2z-g\,(g^2-a^2z),$$
$$h(\theta^2+\phi^2z+2\theta\phi z)=2kc-a\,(g^2-a^2z).$$

Besides these, a seventh formula is needed, giving B′ in terms of the other quantities. The values of B, B′, B″, and B‴ are indicated, but not expressed. They are, in fact, complicated functions of the other quantities, and would each contain, if written at length, from twenty to thirty terms. Mr. Young now calls attention to the general fact that it is always possible to eliminate five of the unknown quantities from his six equations, leaving a single equation of some degree (presumably of a very high degree, except in cases solvable algebraically), from which the numerical value of the remaining unknown quantity can be ascertained. He has therefore succeeded in showing the correct form of the roots of the general quintic, and in proving that it is not impossible to determine and exhibit the roots in that form. He has not, however, reduced the method to a manageable process, except for a single special case of unusual simplicity ($a=0$). To supply this deficiency, and to show in detail how the roots may be determined in their normal form, without transformation, by the aid of a resolvent sextic, is the first object of the present paper. In doing this it will be shown that Mr. Young's original equations may be simplified, and two of the four unwieldy formulæ dispensed with. For each of the trinomials

$$x^5+p_4x+p_5=0,\quad x^5+p_3x^2+p_5=0,$$

there will be afforded a choice between two sextics, one a special case of what I call the first resolvent, the other of the second. After discussing these resolvents, I shall close by making some observations on certain classes of solvable quintics.

On the General Quintic.

"By the resolution of a quintic, I mean the expression of its roots in terms of those of its resolvent sextic."* Tried by this standard, Mr. Young's work needs to be continued and completed. He has shown the normal form of the roots fully and satisfactorily, and has proved the possibility of exhibiting them without any other than a linear transformation of the original equation. But

* Cockle, "Resolution of Quintics," *Quarterly Journal*, IV, 5.

though he has shown that such exhibition is possible, by means of the numerical solution of the six equations quoted, it remains a mere theoretical possibility. The elimination of five of the six unknown quantities, if attempted by what might be called main force, would in most cases involve very great labor.

The methods of resolution devised by Lagrange and Vandermonde were too intricate to be practicable. A method of resolving the trinomial $x^5 + 10\delta x^2 + \zeta = 0$ was given in 1860 by Mr. (now Sir James) Cockle, in the paper from which a sentence was quoted in the foregoing paragraph; but it did not give the full form of the root, nor did it apply to the general quintic. Another process was gradually evolved by the labors of Messrs. Cockle and Harley, during a period of perhaps fifteen years, in numerous papers appearing in various journals. This process reached its culmination in a memoir presented by Professor Cayley to the Royal Society on Feb. 20, 1861. Mr. Cockle showed that any quintic for which the "resolvent product" is evanescent is solvable by radicals, and that the general quintic can, by two successive transformations, aided by his resolvent sextic, be brought into that solvable form. The resolvent product (θ) would be represented in Mr. Young's notation by $625 u_1 u_2 u_3 u_4$. Mr. Cockle's sextic determined the numerical value of θ for the trinomial $x^5 + 10\delta x^2 + \zeta = 0$, and enabled him to transform any equation from this form to a solvable form, the trinomial itself being first obtained by a Tschirnhausen transformation. That the second transformation was not essential was shown by Mr. Harley, who employed a modification of Mr. Cockle's sextic (previously made by the latter) in which the unknown quantity is t (say $5a\sqrt{z}$), and proved that the roots of the trinomial are rational functions of t. Merely for the trinomial, t is not more simple than θ; for the general quintic, however, it is much more simple, being, in Mr. Young's notation, $5a\sqrt{z}$ against $625(g^2 - a^2 z)$. Recognizing this great difference, Mr. Cayley calculated for the general quintic the sextic in t, or rather in $10\sqrt{5a\sqrt{z}}$, showing that its roots are rational functions of those of the quintic, and vice versa.* In his own words, "The roots of the given quintic are each of them rational functions of the roots of the auxiliary equation, so that the theory of the solution of an equation of the fifth order appears to have been carried to its extreme limit." The sextic so discovered by him is said, according to Mr. Harley, "to take an extremely simple form; it may, in fact, be regarded as canonical in the theory of equations of the fifth degree."

*I gather this from the brief abstract in the *Proceedings*, not having as yet seen the memoir. See Cockle, "Researches in the Higher Algebra," *Manchester Memoirs*, XV, 131; Harley, "On the Theory of Quintics," *Quarterly Journal*, III, 343.

On the one hand, therefore, we have the auxiliary sextic thus evolved, whose roots are known to be functions of those of the given quintic. On the other hand Mr. Young now shows the form of the roots of the general quintic, with six simultaneous equations from which to extract their values. It only remains to connect these two theories. The steps which I shall indicate are not necessarily the best for this purpose, and I shall be glad if any one improves upon them; but they have at least this merit, that at no point is it necessary to solve an equation of any degree above the first. They therefore make it possible, if any one will take the trouble, to express the roots directly and visibly in terms of those of a resolvent sextic.

As regards notation, I shall follow that of Mr. Young, except where departure seems clearly warranted by the nature of the results. In lieu of his expression for the general equation, $x^5 - 10gx^3 - 20kx^2 + p_4 x + p_5 = 0$, useful to him as somewhat simplifying his formulæ, I shall employ the normal form

$$x^5 + 10\gamma x^3 + 10\delta x^2 + 5\varepsilon x + \zeta = 0.$$

The reader will kindly notice the respective values of the symbols employed for the coefficients. As thus changed, formulæ (2) become

$$u_1 u_4 = -\gamma + az^{\frac{1}{2}}, \quad u_2 u_3 = -\gamma - az^{\frac{1}{2}}. \tag{4}$$

Observing these and (1), and putting $v = a^2 z$, we see that

$$\begin{aligned} u_1^5 u_4^5 &= z^{\frac{1}{2}} a (v^2 + 10v\gamma^2 + 5\gamma^4) - (5\gamma v^2 + 10\gamma^3 v + \gamma^5) \\ &= z^{\frac{1}{2}} \mathrm{F} + \mathrm{G}, \text{ suppose,} \\ &= [\mathrm{B} + \mathrm{B}' z^{\frac{1}{2}} + (hz + hz^{\frac{1}{2}})^{\frac{1}{2}}][\mathrm{B} + \mathrm{B}' z^{\frac{1}{2}} - (hz + hz^{\frac{1}{2}})^{\frac{1}{2}}] \\ &= z^{\frac{1}{2}} (2\mathrm{BB}' - h) + \mathrm{B}^2 + \mathrm{B}'^2 z - hz. \end{aligned}$$

Everything being rational except $z^{\frac{1}{2}}$, we have

$$\mathrm{F} = 2\mathrm{BB}' - h, \tag{5}$$

$$\mathrm{G} = \mathrm{B}^2 + \mathrm{B}'^2 z - hz. \tag{6}$$

From these

$$\left. \begin{aligned} \mathrm{B}' &= \mathrm{B} + \left(\frac{\mathrm{B}^2 e^2 - \mathrm{F}z + \mathrm{G}}{z} \right)^{\frac{1}{2}}, \\ h &= 2\mathrm{BB}' - \mathrm{F}. \end{aligned} \right\} \tag{7}$$

This determines h and B' when v, a, and B are known, since $z = va^{-2}$, and these three quantities are all we need to seek in order to assign the elements of the root as in (1). We thus avoid the long and complex expressions for B' and B'', and shall see that we have also obviated the necessity of determining the values of the quantities θ and ϕ.

Five of Mr. Young's six equations will still be useful to us. These are, after making the changes of notation already indicated, and putting $c=\frac{1}{2}at$, and (merely for shortness) $\gamma^2-v=\text{V}$, $\gamma^2+v=\text{V}'$,

$$\text{V}\varepsilon=\text{V}\gamma^2+3\text{V}v+\gamma\delta^2-\gamma vt^2-4heva^{-1}(\theta^2-\phi^2z), \quad (8)$$

$$\zeta\ =20vt-4\text{B}, \quad (9)$$

$$\text{B}'''=0, \quad (10)$$

$$4hz(\theta^2+\phi^2z+2\theta\phi)=\delta^2+vt^2+4\text{V}\gamma, \quad (11)$$

$$ha^{-1}(\theta^2+\phi^2z+2\theta\phi z)=-\tfrac{1}{2}\delta t-\text{V}. \quad (12)$$

Calculating B and B''' as indicated by him, but shortening the work by employing the letters V and V' as above stated, and also by putting $\text{Q}=\frac{1}{2}\text{V}'(vt^2+\delta^2)+2\gamma\delta vt+2\gamma\text{V}v+\gamma\text{V}\text{V}'$, and $\text{R}=\gamma vt^2+\text{V}'\delta t+2\gamma^2\text{V}+\text{V}\text{V}'+\gamma\delta^2$, we have

$$\text{V}^2\text{B}=heva^{-1}(\theta^2-\phi^2z)(\text{V}'t+2\gamma\delta)-\tfrac{1}{2}\delta\text{Q}+\tfrac{1}{2}vt\text{R}, \quad (13)$$

$$e\text{V}^2\text{B}'''=0=\theta\left[\gamma(vt^2-\delta^2)ae+\text{Q}+a\text{R}\right]-\phi\left[\tfrac{1}{2}\text{V}'(vt^2-\delta^2)e-\text{Q}-va^{-1}\text{R}\right]. \quad (14)$$

What we have to do is to extract in some way from these equations the values of B, v, and a, in terms of the known quantities γ, δ, ε, and ζ; and as B is given by (9) when v and t are known, we may say that the values desired are those of v, t, and a.

Let $\text{P}=\text{V}\gamma^2+3\text{V}v+\gamma\delta^2-\text{V}\varepsilon-\gamma vt^2$. Then, from (8),

$$4heva^{-1}(\theta^2-\phi^2z)=\text{P}. \quad (15)$$

Eliminating B from (9) and (13),

$$4heva^{-1}(\theta^2-\phi^2z)(\text{V}'t+2\gamma\delta)=20\text{V}^2vt-\text{V}^2\zeta+2\delta\text{Q}-2vt\text{R}. \quad (16)$$

Dividing (16) by (15), and clearing the result of fractions, we have this equation in v and t:

$$\text{V}'t\text{P}+2\gamma\delta\text{P}=20\text{V}^2vt-\text{V}^2\zeta+2\delta\text{Q}-2vt\text{R}. \quad (17)$$

Another such equation is needed. Put $\text{M}=\delta^2+vt^2+4\text{V}\gamma$, and $\text{N}=-2\delta vt-4\text{V}v$. We take the following from (11) and (12) respectively:

$$4heva^{-1}(\theta^2+\phi^2z+2\theta\phi)=\text{M}ea, \quad (18)$$

$$4heva^{-1}(\theta^2+\phi^2z+2\theta\phi z)=\text{N}e. \quad (19)$$

Adding (15) respectively to (18) and (19), and dividing the first result by the second,

$$\frac{\theta+\varphi}{\theta+\varphi z}=\frac{\text{M}ea+\text{P}}{\text{N}e+\text{P}}; \quad (20)$$

subtracting instead of adding,

$$\frac{\theta+\varphi z}{\theta z+\varphi z}=\frac{\text{M}ea-\text{P}}{\text{N}e-\text{P}}. \quad (21)$$

Multiplying these equations together,

$$z^{-1}(\text{N}^2e^2-\text{P}^2)=\text{M}^2e^2a^2-\text{P}^2;$$

whence, since $z = 1 + e^2$, and $a^2 z = v$,

$$N^2 = M^2 v - P^2, \tag{22}$$

or

$$(2\delta vt + 4Vv)^2 = (\delta^2 + vt^2 + 4\gamma V)^2 v - P^2, \tag{23}$$

a second equation in v and t.

We have now, in (17) and (23), two equations in v and t. In any case not critical we can, by division or some other usual process, depress these so as to have an equation of the first degree in v or t, for use in case one of them is known; and we can eliminate either v or t, thus obtaining a resolvent sextic, as will be seen further on. With numerical coefficients, no difficulty will occur, nor is it necessary to frame a formula for t in terms of v. The problem may be considered solved, therefore, as regards v and t, and it only remains to determine a. For this we need an equation of the first degree, not containing θ, ϕ, h, z, or e.

From (14), (20), and (21), the following equations are readily obtained, remembering that $v = a^2 z$ and $z = 1 + e^2$:

$$\frac{\theta}{\varphi} = \frac{Te - Q - Rva^{-1}}{Wae + Q + Ra} = \frac{Mva^{-1} + Pe - N}{N - Ma} \tag{24}$$

$$= \frac{Mv - Nva^{-1}}{Na + Pae - Mv}. \tag{25}$$

Here $T = \frac{1}{2}V'(vt^2 - \delta^2)$, and $W = \gamma(vt^2 - \delta^2)$. When (24) and (25) are cleared of fractions, respectively, and $va^{-1} - a$ is replaced by ae^2, wherever that combination is observed, we obtain from them these:

$$aep = am - n,$$
$$aeq = an - mv.$$

Here $m = NW - PR - MT$, $n = MWv + PQ - NT$, $p = MQ + NR + PW$, and $q = MRv + NQ + PT$. Eliminating ae, finally,

$$a = \frac{pmv - qn}{pn - qm}. \tag{26}$$

For determining the roots of the general quintic, therefore, the following steps are to be pursued:

First, ascertain v or t by eliminating t or v from (17) and (23), or by employing one of the resolvent sextics known to be producible by such elimination, and then obtain the other of these two quantities by means of the same equations;

Secondly, determine a by (26) and z by $a^2 z = v$;

Finally, ascertain B, h, and B$'$ by (9) and (7), thus obtaining all that is necessary to fill out the root-forms (1). We have still to discuss the possible resolvent sextics.

On the Trinomial $x^5 + 5\varepsilon x + \zeta = 0$.

When $\gamma = \delta = 0$, our formulæ become much simplified. Here $\mathrm{v}' = -\mathrm{v} = v$, $\mathrm{Q} = \frac{1}{2} v^2 t^2$, $\mathrm{R} = -v^2$, $\mathrm{P} = v\varepsilon - 3v^2$, and from (17) and (23), after striking out common factors, and putting $u = 25v$, we have

$$tu - t\varepsilon - \zeta = 0, \tag{27}$$

$$u^2 - ut^4 - 6\varepsilon u + 25\varepsilon^2 = 0. \tag{28}$$

Eliminating u, we have this simple sextic in t:

$$\varepsilon t^6 + \zeta t^5 - 20\varepsilon^2 t^2 + 4\varepsilon\zeta t - \zeta^2 = 0. \tag{29}$$

Eliminating t, on the other hand, we have this simple sextic in u:

$$(u - \varepsilon)^4 (u^2 - 6\varepsilon u + 25\varepsilon^2) - u\zeta^4 = 0. \tag{30}$$

When by either of these we ascertain the value of u or t, the value of the other is given by (27), and that of v by $u = 25v$. To illustrate these sextics, we may take the equation $x^5 + \frac{625}{4} x + 3750 = 0$, whose elements are given in detail by Mr. Young. Here $\varepsilon = \frac{125}{4}$, and $\zeta = 3750$. Then by (29) $t = 5$, or by (30) $u = \frac{3125}{4}$; and these values satisfy (27). The following are other forms of (30):

$$(u^3 - 5\varepsilon u^2 + 15\varepsilon^2 u + 5\varepsilon^3)^2 = u(\zeta^4 + 256\varepsilon^5), \tag{31}$$

$$u^6 - 10\varepsilon u^5 + 55\varepsilon^2 u^4 - 140\varepsilon^3 u^3 + 175\varepsilon^4 u^2 - (\zeta^4 + 106\varepsilon^5)u + 25\varepsilon^6 = 0. \tag{32}$$

If we put $u = \varepsilon y$, equation (30) takes this form:

$$(y - 1)^4 (y^2 - 6y + 25) = y\varepsilon^{-5}\zeta^4. \tag{33}$$

In this compact equation we have the unknown quantity expressed directly as a function of the parameter $\varepsilon^{-5}\zeta^4$. Were tables for such purposes worth calculating, a table of single entry might be constructed with great ease, showing a value of the parameter for each value of y, and therefore the converse, which would supply all that is needed to exhibit the roots of the trinomial in their normal form.

These various results are doubtless new, except that (31) may be readily derived from Mr. Cayley's sextic. If we take the square roots of both sides of (31), the result is

$$u^3 - 5\varepsilon u^2 + 15\varepsilon^2 u \mp \sqrt{(\zeta^4 + 256\varepsilon^5)}\sqrt{u} + 5\varepsilon^3 = 0, \tag{34}$$

twin sextics for finding $\pm\sqrt{u}$. If in Mr. Cayley's sextic (I have not seen it, but

the fact cannot be otherwise) we were to put $\alpha = 1$, $\beta = \gamma = \delta = 0$, we should reproduce (34), except that the unknown quantity ($\sqrt{u}$) would be replaced by another equal to $\sqrt{u} \times 2\sqrt{5}$. Equation (31) is, in fact, a special case of a general resolvent in u which, as we shall see hereafter, may be immediately derived from Mr. Cayley's sextic. Similarly, equation (29) is a special case of a resolvent in t.

For determining a by means of (26), we find that, when $\gamma = \delta = 0$, $\mathrm{M} = vt^2$, $\mathrm{N} = 4v^2$, $\mathrm{T} = \frac{1}{2}v^2t^2$, and $\mathrm{W} = 0$; so that, omitting the factor v^3, $m = \varepsilon - 3v - \frac{1}{2}t^4$, $n = \frac{1}{2}\varepsilon t^2 - \frac{7}{2}vt^2$, $p = \frac{1}{2}t^4 - 4v$, and $q = \frac{1}{2}\varepsilon t^2 - \frac{1}{2}vt^2$. For determining h and B' by means of (7), we have $\mathrm{F} = av^2$, $\mathrm{G} = 0$.

On the Trinomial $x^5 + 10\delta x^2 + \zeta = 0$.

When $\gamma = \varepsilon = 0$, we have $\mathrm{V}' = -\mathrm{V} = v$, $\mathrm{Q} = \frac{1}{2}v^2t^2 + \frac{1}{2}v\delta^2$, $\mathrm{R} = v\delta t - v^2$, $\mathrm{P} = -3v^2$, and putting $u = 25v$ as before, we readily obtain from (17) and (23) the following:

$$25\delta^3 - \delta ut^2 = u\zeta - u^2t, \tag{35}$$

$$(25\delta^2 - ut^2)^2 = u^3 - 16\delta u^2 t. \tag{36}$$

Squaring (35), multiplying (36) by δ^2, and comparing the two results, we find that

$$u^2t^2 - 2\zeta ut - \delta^2 u + 16\delta^3 t + \zeta^2 = 0. \tag{37}$$

We have in (35) and (37) two quadratic equations in t, and also in u. If we wish to eliminate x from the two equations $ax^2 + bx + c = 0$, $dx^2 + ex + f = 0$, we find that $b(bd - ae)(cd - af) - a(cd - af)^2 - c(bd - ae)^2 = 0$. Eliminating u and t respectively in this manner from (35) and (37), we have at once these two sextics:

$$16\delta^3t^6 + \zeta^2t^5 - 50\delta^2\zeta t^4 + 40\delta^4t^3 + 15\delta\zeta^2t^2 + 59\delta^3\zeta t - \zeta^3 + 25\delta^5 = 0, \tag{38}$$

$$u^6 - 20\delta\zeta u^4 - 160\delta^4u^3 + 100\delta^2\zeta^2u^2 - (1856\delta^5\zeta + \zeta^4)u + 6400\delta^8 = 0. \tag{39}$$

Here (38) is a special case of the second resolvent, that in t before spoken of, and is new; (39) is a special case of the first resolvent (that in u), immediately to be discussed, and is not new, being the well-known sextic of Sir James Cockle. This equation has had an interesting history. For years the focus of acute researches, when this trinomial was considered as affording the best path to the solution of the quintic, it was at length set aside, and its place was taken by Mr. Cayley's sextic, or rather by that special case of the latter which belongs to this trinomial. It now appears again as one of the most direct aids in the resolution of the equation to which it refers. As announced by Mr. Cockle, our u was represented by $-\frac{1}{25}\theta$, δ by $-\frac{1}{2}\mathrm{Q}$, and ζ by E. He also exhibited it in another

form, to which this corresponds:

$$(u^3 - 10\delta\zeta u - 80\,\delta^4)^2 = (3456\delta^5\zeta + \zeta^4)\,u. \tag{40}$$

By taking the square roots of both sides of this equation, he produced another which had as its unknown quantity what corresponded to $5\sqrt{u}$, and which was afterwards in substance generalized by Mr. Cayley as already stated.

On the First Resolvent.

I have repeatedly referred to two resolvent sextics, the numerical solution of one of which gives the value of u,* leading directly, under Mr. Young's theory as herein extended, to the determination of all the roots of the general quintic in their normal and complete form. This first resolvent, in what seems to be its most convenient shape, is as follows:

$$\begin{aligned}25\,[\tfrac{1}{5}u^3 &- (3\gamma^2+\varepsilon)\,u^2 + (15\gamma^4 - 2\gamma^2\varepsilon + 8\gamma\delta^2 - 2\delta\zeta + 3\varepsilon^2)\,u - 25\gamma^6 + 35\gamma^4\varepsilon \\ &- 40\gamma^3\delta^2 - 2\gamma^2\delta\zeta - 11\gamma^2\varepsilon^2 + \gamma\zeta^2 + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3]^2 \\ = 8u\,[&432\gamma^5\zeta^2 - 1440\gamma^4\delta\varepsilon\zeta + 800\gamma^4\varepsilon^3 + 640\gamma^3\delta^3\zeta - 400\gamma^3\delta^2\varepsilon^2 - 180\gamma^3\varepsilon\zeta^2 \\ &+ 330\gamma^2\delta^2\zeta^2 + 560\gamma^2\delta\varepsilon^2\zeta - 320\gamma^2\varepsilon^4 - 1260\gamma\delta^3\varepsilon\zeta + 720\gamma\delta^2\varepsilon^3 - 15\gamma\delta\zeta^3 \\ &+ 20\gamma\varepsilon^2\zeta^2 + 432\delta^5\zeta - 270\delta^4\varepsilon^2 + 45\delta^2\varepsilon\zeta^2 - 80\delta\varepsilon^3\zeta + 32\varepsilon^5 + \tfrac{1}{8}\zeta^4].\end{aligned} \tag{41}$$

From this equation, when $\gamma = \delta = 0$, we have (31) as a special case, and when $\gamma = \varepsilon = 0$, (40).

That such a sextic may exist, as a result of eliminating t from (17) and (23), is evident upon observing the special cases just mentioned. The literal part of the coefficients might be written out with ease, having regard to the necessary weight of each, but the determination from those equations of the numerical part, by elimination or otherwise, would be a work of much labor. It is unnecessary, as we have seen, since the unknown quantity (ϕ) in Mr. Cayley's sextic is equal to $2\sqrt{(5u)}$. To obtain our first resolvent, therefore, we have only to substitute the latter value in that sextic in lieu of ϕ, to put, with Mr. Young, $\alpha = 1$ and $\beta = 0$, these being the coefficients of x^5 and x^4, and, bringing the irrational terms to one side of the equation, to square both sides. We are, in fact, reversing for the general equation the step taken for the trinomial when Mr. Cockle's θ was replaced by a multiple of its square root, though in a direction somewhat different. While θ differs, as regards the trinomial, from u only by a numerical factor, it is otherwise for the general equation; since $u = 25\alpha^2 z$, while $\theta = 625\,(\gamma^2 - \alpha^2 z)$, the two differing only by a factor when $\gamma = 0$, as happens in the trinomial.

Mr. Cayley's sextic involves a cyclic function of the roots, namely,

$$\phi = r_1r_2 + r_2r_3 + r_3r_4 + r_4r_5 + r_5r_1 - r_2r_4 - r_4r_1 - r_1r_3 - r_3r_5 - r_5r_2.$$

To demonstrate our first resolvent more directly than by employing that sextic would require the building up of a sextic having for its roots $\frac{1}{20}\phi^2$ and the five similar cyclic functions of the roots of the quintic. There is no reason for this being done, unless it should happen, what is hardly probable, that the coefficients of u could be determined more easily than those of ϕ. This question naturally arises: should we not dispense with the symbol u, and with the modified resolvent (41), and rely wholly on ϕ and Mr. Cayley's sextic? Several reasons appear conclusive against this course.

1. Mr. Young's whole theory is based on the employment of symbols having rational values only. Now u is rational, and ϕ is not; $u = 25a^2z$, $\phi = 10\sqrt{5a}\sqrt{z}$.

2. The connection of the resolvent with, and its derivation from, the two equations in v and t would be obscured by the introduction of the symbol ϕ, and the same might be said of the whole theory.

3. As $u = \frac{1}{20}\phi^2$, the use of ϕ would add needless numerical complication to all the processes.

4. Since the value of ϕ is only needed in the form of ϕ^2, it would be a roundabout proceeding to determine first the irrational quantity ϕ by the original sextic, and then to square it, when u can be found directly by our resolvent (41).

5. One of the coefficients in the original sextic is irrational; those of (41) are all rational.

6. The coefficient of ϕ in the original sextic is, as Mr. Cayley states, the square root of the discriminant of the quintic, multiplied by a numerical factor (irrational); the coefficient of u in (41) is that discriminant itself, unchanged.

The values of the coefficients in (41) were taken by me, with the necessary changes, from a translation in the *Analyst* (1877) of a pamphlet published in 1861 by Mr. Adolf von der Schulenburg,* which contained a sextic for $\frac{1}{2}\phi$, which I presume to be identical with that given early in the same year by Mr. Cayley. The coefficient of u has been verified by Dr. Salmon's expression for the discriminant.

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* *Auflösung der Gleichungen Fünften Grades*, translated by Professor A. B. Nelson. The formulæ given for the roots are erroneous.

On the Second Resolvent.

When $\gamma = 0$, the resolvent in t, otherwise complex and as yet uninvestigated, appears as a sextic of remarkable properties, and decidedly preferable in all respects to that in u. In this case, when we eliminate v from (17) and (23), we find that $v\text{A} = -\text{C}$, where A is the quintic in $-t$, and C its canonizant, namely, $\text{A} = -t^5 + 10\delta t^2 - 5\varepsilon t + \zeta$, and $\text{C} = \delta^2 t^3 - \delta\varepsilon t^2 + (\varepsilon^2 - \delta\zeta)t - \delta^3$. This simple relation gives the value of v when that of t has been obtained by means of the second resolvent; the terms of which, ascertained by the same process of elimination, are

$$\delta^3\text{A}^2 + (\zeta + \varepsilon t + \delta t^2)\,\text{AC} + 25t\text{C}^2 = 0, \qquad (42)$$

or, in full,

$$\begin{aligned}&(16\delta^4 + 2\delta\varepsilon\zeta - \varepsilon^3)t^6 + (\delta\zeta^2 - 44\delta^3\varepsilon - \varepsilon^2\zeta)t^5 + (75\delta^2\varepsilon^2 - 50\delta^3\zeta)t^4\\ &+ (40\delta^5 - 40\delta\varepsilon^3 + 30\delta^2\varepsilon\zeta)t^3 + (15\delta^2\zeta^2 - 55\delta^4\varepsilon - 30\delta\varepsilon^2\zeta + 20\varepsilon^4)t^2\\ &+ (3\delta\varepsilon\zeta^2 - 20\delta^3\varepsilon^2 + 59\delta^4\zeta - 4\varepsilon^3\zeta)t + 25\delta^6 - 6\delta^3\varepsilon\zeta - \delta\zeta^3 + \varepsilon^2\zeta^2 = 0.\end{aligned}$$

From this, when $\delta = 0$ and $\varepsilon = 0$ respectively, we have (29) and (38) as special cases.

It is obvious that these two resolvents do not comprise all that are possible. If any one of the unknown quantities becomes known, or any relation between them, the quintic becomes solvable, as Mr. Young has pointed out. Hence any auxiliary equation which determines for us any one of the unknown quantities, or any known function of more than one, will serve as a resolvent.

Mr. Young's theory might have been stated differently by him; as, for example, by making his elements u_1, u_2, etc. five times as large. This would make changes necessary throughout, but would not necessarily cause any change in the coefficients of the resolvents.

It will be seen, by reference to (17) and (23), that a resolvent might be found in which the unknown quantity would have the value of vt. Judging from the trinomials, its expression must be complex.

On Certain Solvable Quintics.

Mr. Young has given a method for determining the roots in the critical case $a = 0$; that is to say, in the case $u = 0$, since $u = 25a^2z$. He has not, I think, indicated any criterion by which such quintics can be recognized when met with. We have in the first resolvent the needed criterion: when the terms unaffected by u vanish, $u = 0$. Quoting those terms, we have $u = 0$ when

$$25\gamma^6 - 35\gamma^4\varepsilon + 40\gamma^3\delta^2 + 2\gamma^2\delta\zeta + 11\gamma^2\varepsilon^2 - \gamma\zeta^2 - 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta + 16\delta^4 - \varepsilon^3 = 0. \qquad (43)$$

For examples of this important class of solvable quintics, it is sufficient to refer the reader to Mr. Young's paper.

Sir James Cockle is the discoverer of a criterion covering another large class of solvable quintics, that namely in which his θ (or $25\gamma^2 - u$) vanishes. The criterion so discovered was afterwards recalculated and verified by Mr. Harley, and may be found in his paper on Symmetric Products, in vol. XV of the *Manchester Memoirs*. We can now reproduce its equivalent by putting $u = 25\gamma^2$ in the first resolvent, with this result ($\mathrm{D} =$ discriminant):

$$[1600\gamma^6 - 640\gamma^4\varepsilon + 160\gamma^3\delta^2 - 52\gamma^2\delta\zeta + 64\gamma^2\varepsilon^2 + \gamma\zeta^2 \\ + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3]^2 = \gamma^2\mathrm{D}. \tag{44}$$

The form of this criterion is very different from that given by Mr. Cockle, but the two must be identical. When $\varepsilon = 4\gamma^2$, and $\delta = 0$, the criterion is satisfied, the quintic in question being well-known as the solvable form of De Moivre; when $\gamma = 0$, (44) gives $-2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3 = 0$, equally well known as the solvable form of Euler. The latter, by the way, is also a special case of (43). In the class of cases now under consideration, one or more of the elements u_1, u_2, u_3, u_4, is non-existent. We have here, as in the case $u = 0$, a critical value of u, and our equations need to be replaced in some points by others. The methods of solution given by Mr Harley in the paper referred to seem well adapted to this class of cases.

The condition of equal roots is the vanishing of the discriminant, and it has long been known that the solution by radicals of such quintics is possible. Perhaps we have here as simple a method of solving them as any that has been perfected. The second member of the resolvent (41) is $\mathrm{D}u$. When D vanishes, the quintic has equal roots, and we may obtain the value of u, and therefore, by the method now presented, the values of the roots, from the cubic equation to which (41) is reduced, namely,

$$\tfrac{1}{5}u^3 - (3\gamma^2 + \varepsilon)u^2 + (15\gamma^4 - 2\gamma^2\varepsilon + 8\gamma\delta^2 - 2\delta\zeta + 3\varepsilon^2)u - 25\gamma^6 + 35\gamma^4\varepsilon - 40\gamma^3\delta^2 \\ - 2\gamma^2\delta\zeta - 11\gamma^2\varepsilon^2 + \gamma\zeta^2 + 28\gamma\delta^2\varepsilon - 2\delta\varepsilon\zeta - 16\delta^4 + \varepsilon^3 = 0. \tag{45}$$

No doubt an important class of solvable quintics will be found to appear when $t = 0$, for which the absolute term of the second resolvent will, when placed equal to zero, afford the necessary criterion. If, for example, we have a trinomial $x^5 + 10\delta^2 + \zeta = 0$, where $\zeta^3 = 25\delta^5$, we know that it is a solvable form.

Other criteria for solvable quintics might be derived, if they were worth having, in numbers without limit, by assigning to u and t, in the two resolvents, different values in terms of the coefficients.

It is conceivably questionable whether the use in any way of knowledge derived from a resolvent sextic does not vitiate the strictly algebraic solution, or "solution by radicals" of a quintic. If, as I have no doubt, it is permissible, it becomes something of a question in casuistry how far we should go in multiplying solvable cases. An infinitely large accumulation of solvable quintics would leave none of the other sort, by which to illustrate the fact that the quintic cannot be solved algebraically; yet it would not change the fact.

Procedure in Critical Cases.

For two of the classes of solvable quintics just considered, the general course of procedure sketched in this paper requires modification, owing to the appearance of vanishing fractions or vanishing equations.

When $v = 0$, we know that $a = 0$, since $v = a^2 z = a^2(1 + e^2)$, where e is rational; also, since $at = 2c$, $vt = 0$, and $vt^2 = 4c^2 z$. Equation (23) shows at once that $P = 0$, whence $\quad 4c^2 z = \gamma^3 + \delta^2 - \gamma\varepsilon. \qquad (46)$

To get the value of z, we need to learn that of c. For this purpose we may modify (26) as follows. Let $M = \frac{1}{2}M'$, $N = t^{-1}N'$, $P = t^{-1}P'$, $Q = \gamma^2 Q'$, $R = \gamma^2 R' c a^{-1}$, $T = \gamma^2 T'$, so that, in this case, $M' = 2\delta^2 + 8c^2 z + 8\gamma^3$, $N' = -8\delta c^2 z$, $P' = \delta^3 + 2\delta\gamma^3 - \gamma^2\zeta + 4\delta c^2 z$, $Q' = 2c^2 z + \frac{1}{2}\delta^2 + \gamma^3$, $R' = 2\delta$, and $T' = 2c^2 z - \frac{1}{2}\delta^2$. Then (24) and (25) take this shape:

$$\frac{\theta}{\varphi} = \frac{T'e - Q' - R'cz}{Q' + R'e} = \frac{M'cz + P'e - N'}{N' - M'c} \qquad (47)$$

$$= \frac{M'cz - N'z}{N' + P'e - M'cz}. \qquad (48)$$

When (47) and (48) are cleared of fractions, respectively, and $z - 1$ is replaced by e^2, we have

$$cep' = cm' - n'$$

$$ceq' = cn' - m'c^2 z,$$

where $m' = -P'R' - M'T'$, $n' = P'Q' - N'T'$, $p' = M'Q' + N'R'$, and $q' = M'R'c^2 z + N'Q' + P'T'$; and eliminating ce,

$$c = \frac{p'm'c^2 z - q'n'}{p'n' - q'm'}. \qquad (49)$$

Hence, when $v = a = 0$, we must first determine $c^2 z$ by (46), then c and z by (49), then B by (9), which gives $B = -\frac{1}{4}\zeta$, and B' and h by (7), where in this case $F = 0$ and $G = -\gamma^5$, so that

$$B' = B + \left(\frac{B^2 e^2 - \gamma^5}{z}\right)^{\frac{1}{2}}, \qquad (50)$$

$$h = 2BB'. \qquad (51)$$

In the special case $v = a = c = 0$, it is readily shown that $\sqrt{z} = \dfrac{\varepsilon^2 + \gamma^4}{\varepsilon^2 - \gamma^4}$.

When $v=\gamma^2$, it would no doubt be possible to perfect a method of procedure to take the place of that given by Mr. Harley, but, so far as I can now see, it would not be advantageous. In any given case of this sort, when desired, the roots can be shown in Mr. Young's form. Thus, for example, if $z=1$, $t=0$, and $a=-\gamma$, we have DeMoivre's case already cited. Here $\text{B}=-\frac{1}{4}\zeta$, by (9); and we readily find by (7), since $\text{F}z=\text{G}=-16\gamma^5$, and $e=0$, that $\text{B}'=\text{B}$, and $h=\frac{1}{8}\zeta^2+16\gamma^5$.

Milwaukee, March 7, 1884.

Postscript, *March* 24.—A different form for the roots, probably new,* may be obtained very readily. Let $l_1=u_1u_4$, and $l_2=u_2u_3$; also let $m_1+n_1=u_1^2u_3$, $m_1-n_1=u_4^2u_2$, $m_2+n_2=u_2^2u_1$, and $m_2-n_2=u_3^2u_4$. Then $m_1^2-n_1^2=l_1^2l_2$, and $m_2^2-n_2^2=l_2^2l_1$, and since $u_1^5=(u_1^2u_3)^2(u_2^2u_1)(u_2u_3)^{-2}$, and similarly for u_2^5 and the rest, we have at once

$$\left.\begin{array}{ll} u_1^5=(m_1+n_1)^2(m_2+n_2)l_2^{-2}, & u_2^5=(m_2+n_2)^2(m_1-n_1)l_1^{-2}, \\ u_3^5=(m_2-n_2)^2(m_1+n_1)l_1^{-2}, & u_4^5=(m_1-n_1)^2(m_2-n_2)l_2^{-2}. \end{array}\right\} \tag{52}$$

Here, following our notation, it will be seen that $l_1=-\gamma+\surd v$, $l_2=-\gamma-\surd v$, $2m_1=-\delta+t\surd v$, $2m_2=-\delta-t\surd v$, and, on determining the proper signs for the radicals by reference to a special case, that $n_1=-\surd(m_1^2-l_1^2l_2)$, and $n_2=\surd(m_2^2-l_2^2l_1)$. Thus when v and t are ascertained, we have the roots of the general equation expressible in terms of these quantities, or of one of them; in this way fully satisfying Sir James Cockle's definition, already quoted, of the problem of the Resolution of the Quintic.

I have hitherto left the fundamental equations (17) and (23) unexpanded, believing them in that form better adapted for use in numerical calculations. I find, however, that they both contain the factor $v-\gamma^2$, and that when this factor is expelled they may be written in full as follows, each being given in two forms, arranged respectively according to the powers of v and t:

$$\left.\begin{array}{l} 25tv^2-(\gamma t^3+\delta t^2+\varepsilon t+10\gamma^2t+\zeta)v+\gamma\text{H}t+\text{K}=0, \\ \gamma vt^3+\delta vt^2-(25v^2-10\gamma^2v-\varepsilon v+\gamma\text{H})t+\zeta v-\text{K}=0; \end{array}\right\} \tag{17$_a$}$$

* The German writer Schulenburg, already referred to, would have reached this form in substance had he not made an error in one of his equations, which was perpetuated in those which followed. Any critic may have noticed this, and so found the true form. But his whole view of the subject was different from the present, nor did he even attempt to show how to determine the quantities corresponding to our *t*.

$$\left.\begin{aligned} &25v^3-(t^4-14\gamma t^2+16\delta t+35\gamma^2+6\varepsilon)v^2+(2\mathrm{H}t^2+2\gamma\delta^2\\ &\qquad\qquad +4\gamma^2\varepsilon+11\gamma^4+\varepsilon^2)v-\mathrm{H}^2=0,\\ &v^2t^4-(14\gamma v^2+2\mathrm{H}v)t^2+16\delta v^2t-25v^3+35\gamma^2v^2+6\varepsilon v^2\\ &\qquad\qquad -2\gamma\delta^2v-4\gamma^2\varepsilon v-11\gamma^4v-\varepsilon^2v+\mathrm{H}^2=0.\end{aligned}\right\}\qquad(23_a)$$

Here $\mathrm{H}=\gamma^3-\gamma\varepsilon+\delta^2$, and $\mathrm{K}=\gamma^2\zeta-2\gamma\delta\varepsilon+\delta^3$. By subtracting $(23_a)\times t$ from $(17_a)\times v$, there results this second quadratic in v:

$$\begin{aligned}&(t^5-15\gamma t^3+15\delta t^2+25\gamma^2t+5\varepsilon t-\zeta)v^2\\ &\qquad -(2\mathrm{H}t^3+\gamma\delta^2t+5\gamma^2\varepsilon t+10\gamma^4t+\varepsilon^2t-\mathrm{K})v+\mathrm{H}^2t=0.\end{aligned}\qquad(53)$$

If we represent (17_a) by $av^2+bv+c=0$, and (53) by $dv^2+ev+f=0$, we find on depressing them that $v=\dfrac{cd-af}{ae-bd}=\dfrac{bf-ce}{cd-af}$, whence

$$(cd-af)^2+(bd-ae)(bf-ce)=0. \qquad (54)$$

Dividing this, after expansion, by $-\gamma^3t^4-\gamma^2\delta t^3+(16\gamma^4+\gamma\delta^2-2\gamma^2\varepsilon)t^2+(\delta^3-2\gamma\delta\varepsilon)t$, we obtain at once the second resolvent, represented by (42) when $\gamma=0$. This important sextic proves to be of the simplest possible nature, being in short

$$5^2\mathrm{C}^2-\mathrm{AL}=0, \qquad (55)$$

where A is the quintic in $-t$, namely, $-t^5-10\gamma t^3+10\delta t^2-5\varepsilon t+\zeta$, L its simplest linear covariant, and C its canonizant. That is to say, $\mathrm{L}=l_0+l_1t$, and $\mathrm{C}=c_0+c_1t+c_2t^2+c_3t^3$, where $l_0=9\gamma^4\zeta-20\gamma^3\delta\varepsilon+10\gamma^2\delta^3+8\gamma^2\varepsilon\zeta-12\gamma\delta\varepsilon^2-2\gamma\delta^2\zeta+6\delta^3\varepsilon+\delta\zeta^2-\varepsilon^2\zeta$, $l_1=15\gamma^4\varepsilon-10\gamma^3\delta^2+2\gamma^2\delta\zeta-14\gamma^2\varepsilon^2+22\gamma\delta^2\varepsilon-\gamma\zeta^2-9\delta^4+2\delta\varepsilon\zeta-\varepsilon^3$, $c_0=2\gamma\delta\varepsilon-\gamma^2\zeta-\delta^3$, $c_1=\gamma\delta^2-\gamma^2\varepsilon-\delta\zeta+\varepsilon^2$, $c_2=-\gamma^2\delta+\gamma\zeta-\delta\varepsilon$, and $c_3=\gamma^3-\gamma\varepsilon+\delta^2$. After these quantities are calculated, the resolvent (55) may be used in this expanded form,

$$\begin{aligned}&(l_1+25c_3^2)t^6+(l_0+50c_2c_3)t^5+5(2\gamma l_1+5c_2^2+10c_1c_3)t^4+10(\gamma l_0-\delta l_1+5c_0c_3+5c_1c_2)t^3\\ &\quad+5(-2\delta l_0+\varepsilon l_1+5c_1^2+10c_0c_2)t^2+(5\varepsilon l_0-\zeta l_1+50c_0c_1)t-\zeta l_0+25c_0^2=0.\end{aligned}\qquad(56)$$

I have followed Mr. Young in treating the quintic without its second term, but it will be seen that neither of the resolvents is thus limited in its nature.

Note.—On page 314 of the present volume of this *Journal* I gave formulae (No. 52) for the elements of the roots of the general equation of the fifth degree. In practice, these formulae give the roots of two conjugate equations, according to the sign taken for $\sqrt{v}$. A change in the sign causes an interchange between l_1 and l_2, m_1 and m_2, n_1 and n_2. If the formulae fail at first to give the root of any given equation, it is only necessary to make these interchanges, or (the same thing in final effect) to change the sign of n_1 or of n_2.

When v and t have been determined for any quintic, we can at once construct the conjugate quintic. Equation (23_a) is a quadratic in ε, one value being the given value of ε, the other being that of ε in the conjugate quintic. From the latter value, that of ζ may be obtained by (17_a). For example, if the given quintic be $x^5+\frac{625}{4}x+3750=0$, the conjugate is $x^5+\frac{3125}{4}x+3125=0$, and a root of the latter is given by (52) after making the interchanges referred to, or after changing the sign of n_1 or of n_2.

April 5, 1884.

Some Papers on the Theory of Numbers.

By Arthur S. Hathaway, *Johns Hopkins University.*

The principles upon which the following papers are founded were developed while attending the Lectures of Professor Sylvester on the Theory of Numbers, in 1879–80; and were presented before the Mathematical Seminary in May, 1880, Jan., Feb., March, 1881. The death of my wife, who had shared with me in the work, and the professional duties of a stenographer, left neither opportunity nor inclination to make a more extended publication of them. Feeling confident, however, that the principles were of value in simplifying the Theory of Numbers, I returned to the subject in the beginning of the present year (1884). It is due to Professor Sylvester to state that the beginnings of my knowledge in the Theory of Numbers were obtained entirely from short-hand notes of his lectures; and that it was his suggestive presentation of the Theory of Congruences that led to the development of these principles.

The first paper, while it is introductory to, is intended to form a part of the second paper. The latter paper is founded upon an extension of the idea of division, which permits the consideration of any dividend whatever. Subjects which are ordinarily considered only in the general theory of Ideal Primes can thus be treated much more simply; and so naturally is one led to the necessary treatment, that I arrived at the solutions of the principal problems of these subjects without any knowledge of the theory of Ideal Primes. An example is the resolution of cyclotomic functions with respect to a prime modulus not only, which lies at the foundation of Kummer's theory of Ideal Primes, but with respect to a power of a prime.

I.—On Integer Classes.

There is a marked similarity between the ideas of Divisibility and Evanescence, which is exemplified in the three fundamental principles of division, viz.:

1st. If two terms are separately divisible, so is their sum and difference.

2d. If a term is divisible, so is a multiple of it.

3d. If a product is divisible and one factor is neither divisible nor *partially* divisible (*i. e.* relatively prime to the divisor), then the remaining factor is divisible.

The last principle is of the same utility in the Theory of Congruences as is the corresponding one in the Theory of Equations; and what happens in the former theory would have its analogue in the latter if it dealt with *partial* zeros.

The combination of the second and third principles leads to a fundamental proposition connecting different divisors, viz. that if a term is divisible by several relatively prime divisors, it is divisible by their product; while the combination of the first and second leads to another fundamental proposition which furnishes the basis of a method for utilizing the similarity above noticed, viz., any combination of sums and products of terms retains its remainder with respect to a given divisor, no matter how the several terms of the combination be replaced by others having the same remainders. That is to say (grouping together as a *class* all terms which have the same remainder), the results of any combination of sums and products of classes constitute members of one and the same class. Here we mean, for example, by the product of two classes, the assemblage of all possible products between the terms of the two. We are thus led to consider a class as a molecular body, as it were, any portion of which identifies the whole.

What corresponds to division among the classes is the theory of the linear congruence, in the ordinary sense, or as it will be here termed, of the linear modular equation $AX=B$. The problem for solution is a special one, limiting us to the question whether any of the given classes satisfy this modular equation, which is not answered by the symbolic solution $\frac{B}{A}\cdot$ The performance of the division here indicated might be presumed, in general, not to give rise to integers; although if, and only if there are integers resulting, is there a solution in the restricted sense which confines us to the consideration of integers only. The application of the third principle to the determination of the maximum number of solutions (by the substitution of two solutions, precisely as in the Theory of Equations) naturally excludes from general consideration the case where A is a divisible or partially divisible class (or, as it might legitimately be called, a *zero*, or *partially zero* class). With this exclusion, linear modular equations are shown to have but one solution. That the solution is not merely symbolic, but actually one of the given classes, is due to the distinctive feature that the number of classes is finite, and not infinite, so that AX, A multiplying successively all the

classes X, and giving rise each time to different classes B, must repeat all the classes over again. There are several important consequences of this feature.

First, is the linear relation among any relatively prime integers, $k_1, k_2, \ldots$, that the equation

$$\frac{k_1 k_2 \ldots}{k_1} x_1 + \frac{k_1 k_2 \ldots}{k_2} x_2 + \ldots = 1$$

is soluble in integers. Second, is the property that successive involutions of a class must finally bring it into some class whose powers repeat itself. These, which have been called repetents,* are *zero*, *unity*, and, for composite moduli, various *partial zeros*. Third, is the existence of cycles of classes, defined by the property that the members of a cycle are repeated over again in some order when severally multiplied by any one member of the cycle. An important example is the cycle composed of all classes which are relatively prime to the modulus.

It is worthy of note that this definition secures the property within the cycle, that quotients are one valued without reference to the restriction of the third principle; and it would be perfectly proper to treat cycles upon this supposition. The properties of cycles may be developed, however, from the definition alone, *ab initio*, whatever the nature of the terms with which we deal. The following are a few of the immediate deductions from the definition, the more general properties being reserved for a future paper:

A cycle contains all the powers of its members, and, therefore, some repetents. If the multiplication of each member of a cycle by a given term leave one member unaltered, it will leave all unaltered; for we may first exhibit (by multiplication) the unaltered member as a factor of all, and then perform the multiplication. If the multiplier is a member it does not alter itself, and is therefore a repetent. Any member raised to the power represented by the number of members is a repetent; for it leaves the product of all the members (which is a member) unaltered. All the repetents of a cycle are equal, since either of two is the same as their product. Every member is repeated as many times as is the repetent, and conversely; for, multiplying the terms of a cycle by a member exhibits it wherever there is a repetent; and multiplying by a power of the member which is one less than the number of members, exhibits the repetent wherever the member occurs.

* By Mr. Mitchell, in his article on Binomial Congruences, in Vol. III of this Journal.

A fourth consequence of this feature is a *form* into which a class may be thrown, which exhibits its relations to the classes which it determines with respect to the relatively prime factors of the modulus. It is well known that any class $Y(\text{mod.}\ k_1k_2\ldots)$ can be thrown, in one way, and one way only, into the form $\frac{k_1k_2\ldots}{k_1}X_1+\frac{k_1k_2\ldots}{k_2}X_2\ldots$, where $k_1, k_2, \ldots$ are relatively prime factors of the modulus; viz. X_i is the class (mod. k_i) which is the solution of $\frac{k_1k_2\ldots}{k_i}X_i = Y(\text{mod.}\ k_i)$. Denoting by Y_i the class determined by $Y(\text{mod.}\ k_i)$, and writing $h_i=\frac{k_1k_2\ldots}{k_i}$, the symbolic solution for X_i is $\frac{Y_i}{h_i}$; the use of which in the above form is legitimate, if we consider it in the restricted sense of denoting only the integral terms which occur in performing the indicated division. We may thus write:

$$Y = h_1\frac{Y_1}{h_1}+h_2\frac{Y_2}{h_2}+\ldots(H).^*$$

The value of this form consists in the fact that any combination of classes (Y), $(\text{mod.}\ k_1k_2\ldots)$ is obtained by forming the same combination of corresponding arguments $(Y_i)(\text{mod.}\ k_i)$; as is evident on considering the way in which the arguments are derived from the class which the form represents. For the zero class the arguments are zeros; for the partial zeros, corresponding partial zeros; for the unit class the arguments are units; for repetents, the arguments are repetents; and if $k_1k_2\ldots$ each involves a single prime only, the arguments of the repetents are confined to zero and unity. Corresponding arguments of cycles are cycles.

The least power (index) for which λ becomes a repetent is the least common multiple of its indices with respect to the relatively prime factors of the modulus which is great enough to permit the partial zero arguments to become zero. In general, the form (H) renders visible, so to speak, the properties of a class in terms of its simultaneous properties with respect to the relatively prime factors of the modulus; and we see that it may be stated as a general principle that the number of classes which possess a property whose negation among the arguments

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* Writing $h_i\frac{1}{h_i}=R_{\bar{k}_i}$ we obtain a form which is given by Mr. Mitchell, $R_{\bar{k}_i}$ being his notation for the common solution of $x\equiv 0$, mod. h_i, $x\equiv 1$, mod. k_i. This common solution satisfies the definition of a repetent, $x^2\equiv x\ (\text{mod.}\ k_1k_2\ldots)$ and also $R_{\bar{k}_i}R_{\bar{k}_j}\equiv 0\ (\text{mod.}\ k_1k_2\ldots)$; whence the property of the form $R_{\bar{k}_1}Y_1+R_{\bar{k}_2}Y_2+\ldots$ In the product of two such forms, for example, the cross products drop out, leaving only the products of corresponding arguments multiplied by the squares of the corresponding coefficients, which are repetents. It appears that a form possessing the same property is obtained by merely writing the symbolic solution for the variables X_i in a well-known form, which at once discloses the repetents and their properties.

involves its negation in the class, is the product of the number of classes possessing that property with respect to each of the relatively prime factors of the modulus. As examples are: the property of being relatively prime to the modulus; of containing the factors common to the modulus and a given number; of being n^{th} power residues of the modulus; of satisfying a given congruence with respect to the modulus.

What corresponds to evolution among the classes is the resolution of the Binomial congruence $X^n = A$. The n^{th} root of a class A gives rise in general only to surds; and if, and only if exact n^{th} powers occur in the class A, will there be a solution. Having found one solution, there is the same dependence upon the solutions of $X^n = 1$ for the others as in the Theory of Equations. The determination of the solvable equations of this sort constitutes the theory of n^{th} power residues; and for this, special methods have been invented, having no analogy to methods in the Theory of Equations; for example, the Theorems of Reciprocity.

So far as the theory of the general equation of the n^{th} degree is concerned, the substitution of $n+1$ solutions and elimination of coefficients, leads, precisely as in the Theory of Equations, to the conclusion that the product of the differences of $n+1$ solutions multiplied into each coefficient is a divisible quantity, so that for a prime modulus, the number of solutions cannot exceed the degree unless the modular equation is identical through the divisibility of each of its coefficients. The reciprocal of a solution cannot contain any factor of the modulus if the coefficient of X^n does not; so that modular equations which have no impossible or indeterminate solutions can be reduced to the standard form $X^n + A_1 X^{n-1} + \ldots A_n = 0$. In the further theory, we catch glimpses, for the most part as the result of special and disconnected methods, of similarity to the Theory of Equations. Equal roots $(X - A)^2$, for example, render the discriminant divisible, and common roots, the resultant. In the case of a prime modulus, Galois' Theorem furnishes a basis for what are called imaginary solutions, giving to a modular equation its complement of solutions. Investigators, however, have kept constantly before them the restriction that they were dealing with integers; a self-imposed restriction which is in nowise indicated by the fundamental principles of division, which only require that the *divisor* shall be an integer. This restriction is so vital that were it not for the *accident* of the theory that the number of integer classes is finite, even a linear equation* would not have, in general, a solution. We may illustrate

this by reference to algebraic division, the theory of modular equations in which has not been developed. The residues of an integer algebraic function, though less than it in degree, are infinite in number.

In division and evolution, the fractions and surds were neglected, because there was no place for them. A true notion of division will give the fractions and integers which result from the division of one class by another a place together. The n surd n^{th} roots, and in general, the n roots of an algebraical equation of the n^{th} degree will have a place in n classes, connected by the same relations as the roots themselves. Some will fall in the integer classes; others will form new classes which contain no integers.

II.—General Theory of Division.

Having given a system of integers consisting of primes, their powers and products,* the idea of division confines us, in the first instance, to the system. But by confining our divisors to that system, the fundamental principles of division will permit the consideration of any terms whatever (roots of equations in the system) as dividends—the quotients being restricted, of course, not to involve the reciprocals of any divisors of the divisor.† The factors of a divisor are the partial zeros of its modular system. The reciprocals of these factors are the partial infinites; and unless they be excluded from general consideration, any quantity whatever can be made to contain the divisor.

On grouping together as a class all terms which have the same remainder, the integral system will fall either into an infinite or a finite number of classes; and there is a marked difference between the theory of division in these two cases. For example, if the number of classes is infinite, the results of division

* It is necessary for the existence of the first principle that the integers of the system should repeat themselves by addition and subtraction; and for the existence of the third principle that the resolution of an integer into products of powers of primes should be possible in only one way.

† While this notion of division is sufficiently definite so long as the dividend is an integer or a fraction, when we introduce irrational dividends, division by a prime say, is not exactly defined without regarding a root of the prime as a factor of it. The actual exhibition of an integer as a factor of an irrational quantity may be practically difficult. Thus, a root of $x^2 - 3x + 5$ is divisible by 5, a prime of the ordinary system, whose only factors are roots of 5; but we can substantiate this only theoretically, viz., the product contains 5, and since the square of the difference is relatively prime to 5, one root cannot contain $\sqrt[n]{5}$ and the other $\sqrt[m]{5}$. This conception of division which renders it no more difficult, theoretically, to consider irrational than rational quantities in the Theory of Numbers, was forced upon me, in developing the analogy between equations and congruences. I had defined $f^{-1}(a)$ as a solution of $f(x) \equiv a$ mod. k, and shown that its ordinary combinations also held (mod. k); also that $f^{-1}(a + \lambda k)$ was a solution, and gave all the integral solutions. Then defining $f^{-1}(a)$ as imaginary (mod. k), if variations of a by λk gave no integers, I was led to define k times an imaginary quantity as $\equiv 0$ mod. k.

between two classes will not, in general, involve integers, the quotients being entirely fractional. Only if there exist among relatively prime factors of the modulus the relation that $h_i x + k_i y = 1$ is soluble in integers, may every class be thrown into the form (H). As in this case, there exist repetents other than zero or unity, the modulus must be a divisor of one or more terms of the form $x^2 - x$, where x is its residue. There is no modular equation of finite degree which is satisfied for every integral class and only for such, by means of which we may satisfy ourselves whether a root of a given equation determines an integral class or not; but if a modular equation be written in the standard form, and the solution is integral, the coefficients must be integral, and the solution a divisor of the constant term.

There is yet an unsettled question in the extension of the notion of division to any dividend whatever, viz., what is the relation as to divisibility between the modulus and any of its roots. In other words, what are the roots of the zero class? If we assume them to be the zero class itself, we make no distinction between division by an integer and any of its powers or roots. We practically *group* the powers and roots of a prime together, and the assertion that a given dividend contains k, merely asserts that it contains some one out of the products of the groups which the prime factors of k determine. We are only sure, for example, if the dividend is integral, that it contains some integral powers of these primes. Observe that we have arranged the system of divisors consisting of primes, their powers and roots into a system consisting of *group* primes $[p]$ and their products, the powers and roots of any one of which repeat itself. With such a system of divisors it is evident that roots of the zero class are zero; and when the modulus is a prime group, the correspondence with the Theory of Equations is exact. The more general supposition is that roots of the zero class are zeros or partial zeros, so that the modular equation $X^n = 0$ does not have all of its roots equal. We thus have the peculiarity of an infinite number of partial zeros for a modulus $p^i q^j$, arranged, however, in a finite number of relatively prime groups, $[p]$, $[q]$ ...

The determination of the arbitrary variable of a function

$$f(x) \equiv x^n + a_1 x^{n-1} + \ldots a_n,$$

so that $f(x)$ shall be divisible by a given modulus k is one and the same thing with seeking the solutions of the infinite number of equations embraced in the modular equation $X^n + A_1 X^{n-1} + \ldots A_n = 0$, where $A_1, \ldots A_n$ are the classes

with respect to the modulus k which are determined by $a_1, \ldots a_n$. For, since these equations differ only by algebraic terms which identically contain k, a root of any one when substituted in any other, $f(x)$, for example, renders it divisible by k; and if β be a quantity which substituted in $f(x)$ renders it divisible by k, then β is a root of one of these equations, viz. $f(x) - f(\beta) = 0$.

We now seek the number of classes which satisfy a modular equation of the n^{th} degree, not by the substitution of $n+1$ solutions, the result of which has been already pointed out, but by a more direct method. The roots of $f(x) = 0$ determine n classes, $X_1 = \alpha_1 + \lambda_1 k$, $X_n = \alpha_n + \lambda_n k$. On the other hand these classes determine a modular equation $(X - X_1) \ldots (X - X_n) = 0$, which includes $f(x) = 0$, and an infinite number of equations which differ from it by identical multiples of k. They are therefore contained in $X^n + A_1 X^{n-1} + \ldots A_n = 0$; but conversely, β being a root of one of these latter equations, it simply renders $f(\beta)$ divisible by k, which may be through distribution of the factors of k among the terms of the product $(\beta - \alpha_1) \ldots (\beta - \alpha_n)$ so that not a single factor shall contain k. The quantity β will not then be found among the classes $X_1 \ldots X_n$ and the first modular equation will not contain all the equations of the latter. We may state the matter in this way: among the equations $f(x) + k\phi(x) = 0$, certain forms of $\phi(x)$ correspond to the equations determined by the roots of one, $f(x) = 0$; and the different sets of such forms correspond to different ways of separating the modular equation into linear modular factors.

As to the possibility of a multiplicity of ways, it depends upon the possibility of finding a quantity β which shall be common to two classes $\alpha_1 + h_1\lambda_1$, $\alpha_2 + k_1\lambda_2$, where h_1, k_1 are relatively prime factors of k; that is to say, upon the possibility of the solution of $h_1\lambda_1 - k_1\lambda_2 = \alpha_2 - \alpha_1$; λ_1, λ_2 not involving the reciprocals of factors in k_1, h_1. This is possible if $h_1 x - k_1 y = 1$ is soluble in integers; and the form (H) then gives the common terms $\beta + \lambda h_1 k_1$ from the arguments consisting of any two solutions of the modular equation with respect to h_1, k_1 as moduli. In case, therefore, the system of integers is such that the number of classes with respect to any modulus is finite, the solution of modular equations is made to depend upon their solution with respect to the relatively prime factors of the modulus.

The number of solutions, as has already been pointed out, is then the product of the number of solutions with respect to each relatively prime factor of the modulus. The like may be shown with respect to the resolution of a function into modular factors, by an extension of the form (H). Thus, if $f_i(X)$

denote the modular function determined by $f(x)$ mod. k_i, then $f(X) = h_1 \frac{f_1(X)}{h_1} + h_2 \frac{f_2(X)}{h_2} + \ldots$ mod. $k_1 k_2 \ldots$ As an example, if $f_1(X) = A.B$ mod. k_1, $f_2(X) = C$ mod. $k_2 \therefore f(X) = \left(h_1 \frac{A}{h_1} + h_2 \frac{1}{h_2}\right)\left(h_1 \frac{B}{h_1} + h_2 \frac{C}{h_2}\right)$

$$= \left(h_1 \frac{B}{h_1} + h_2 \frac{1}{h_2}\right)\left(h_1 \frac{A}{h_1} + h_2 \frac{C}{h_2}\right) \text{ mod } k_1 k_2.$$

Confining our attention now to powers of primes as moduli, the factors of the modulus are its roots; and if these can be distributed among two or more factors $\beta - \alpha_1$, $\beta - \alpha_2$ of $f(\beta)$, which is, as we have seen, the necessary and sufficient condition that $f(X)$ shall be resoluble into linear modular factors in more than one way, then the differences, $\alpha_2 - \alpha_1$ of these factors will also contain a root of the modulus. Therefore the product of all differences, or the discriminant of $f(x)$ will contain some root of the modulus. The discriminants of all the equations of a modular equation belong of course to the same class, since they are rational functions of the coefficients. Therefore, if the discriminant of a modular equation is relatively prime to the modulus, it cannot be resolved in more than one way into linear factors. On the contrary, if the discriminant contain a factor of the modulus, the modular equation is resolvable into linear factors in an infinite number of ways. Suppose, for example, that $\alpha_1 - \alpha_2 = \mu_2 p^{y_2}, \ldots \alpha_1 - \alpha_r = \mu_r p^{y_r}$, where y_2, y_r are quantities greater than zero, arranged, we will suppose, in ascending order of magnitude. Then if $\beta = \alpha_1 + \lambda p^{i_1}$, $f(\beta) = \lambda p^{i_1} (\lambda p^{i_1} + \mu_2 p^{y_2}) \ldots (\lambda p^{i_1} + \mu_r p^{y_r})\, Q$. In order that p^i may be a divisor of $f(\beta)$, it is evident that i_1 cannot be less than $\frac{i}{r}$, since in making up i, we must take the least exponent i_1, y_s in each of the above factors. The suppositions $i_1 = y_s$ are included in $\beta = \alpha_s + \lambda p^{i_s}$. If $\frac{i}{r}$ fall between y_s and y_{s+1}, the least value of i_1 is the first of the series $\frac{i - \sum\limits_{2}^{s'} y_s}{r - s' + 1}$, $s' \mathrel{\overline{>}} s$, which falls between $y_{s'}$ and $y_{s'+1}$, and the same for i_s. Notice that $i_1 = \frac{i}{r}$ if $\frac{i}{r} \mathrel{\overline{<}} y_2$. The values of β which render $f(\beta)$ divisible by p^i are therefore of the forms $\alpha_1 + \lambda p^{i_1} + \mu p^i, \ldots \alpha_n + \lambda p^{i_n} + \mu p^i$; and we call attention to the fact that when the discriminant contains a factor of the modulus, n values of β whose differences are relatively prime to the modulus cannot be found. If we make no distinction between division by the modulus and any power or root of it, these forms (which consist of groups of classes corresponding to variations of the parameter λ) reduce to members of the classes

determined by $\alpha_1, \ldots \alpha_n$, of which those determined by the cognate roots $\alpha_1 \ldots \alpha_r$, for example, are equal. The modular equation is then resolvable into linear factors in only one way, and involves equal factors.

This notion of division, which it is useful to introduce when considering a function with respect to those exceptional divisors of it which are divisors of its discriminant, does not determine the actual divisor to be p^i, but, as already pointed out, only locates it among the group $[p]$, consisting of the powers and roots of p^i. Such an hypothetical division leads, however, to certain definite results with regard to actual division by p. Namely: (1) in the determination of the integer classes which render $f(x)$ divisible by some p^α, we are sure that the integers of those classes render it divisible by p; (2) in establishing that one algebraic function divides another for some p^α as modulus, we are sure that α will be at least $=1$, for, we may proceed by actual division with respect to the modulus p until we get a residue of the divisor, which must identically contain some p^α, and therefore p, since the coefficients are integral.

The general method for determining whether or not the class pertaining to a given quantity is integral, is to substitute it in the one or more modular equations whose solutions are known to be integral, and which include among their solutions every integral class.

The number of integer residues of a prime modulus p being π, we have such a modular equation in

$$X^\pi - X = 0 \ (\text{mod. } p). \qquad (P)$$

For, excluding the zero class, the remaining classes satisfy $X^{\pi-1} = 1$, since they form a cycle; and the differences of all the classes being relatively prime to p, so is the discriminant of (P). The integral classes are therefore the complete solutions of (P).

With respect to p^i as modulus, we have a set of such modular equations,

$$X^\pi - X = Rp \ (\text{mod. } p^i) \qquad (R)$$

where R is an integral class (mod. p^{i-1}). For, by (P), every integer satisfies an equation of the form $x^\pi - x = rp$, and therefore every integral class (mod. p^i) satisfies one or other of the modular equations (R) determined by these equations. Also, each modular equation (R) has only π solutions, since its discriminant is relatively prime to p; and the whole number of such modular equations is $\pi^{i-1} =$ the number of ways of filling $\lambda_1 p^{i-2} + \ldots \lambda_{i-1}$ with residues of p. The π^i integral classes are therefore the complete solutions of (R).

Since the integers, if any, which render $f(x)$ divisible by p^i are obtained, if the discriminant is relatively prime to p, by adding multiples of p^i to the roots of $f(x)=0$, the substitution of these roots in the modular equations (R) determines whether this is possible. If, however, the discriminant contains p, such integers may be obtained by adding to some of the roots multiples of less powers of p; for example, by adding to one of the cognate roots $\alpha_1, \ldots \alpha_r$ multiples of p^{i_1}, where i_1 has a least value less than i and $\overline{>}\frac{i}{r}\cdot$ This renders it necessary to consider the possibility of a quantity α_1 becoming an integer by the addition of (restrictedly) a multiple of a fractional power of p.

This is possible only when α_1 is one of a set of cognate roots of an irreducible integral equation whose discriminant contains p. For, if an integer $\beta=\alpha_1+\lambda p^a$, then $f(\beta)=\lambda p^a(\lambda p^a+\alpha_1-\alpha_2)\ldots(\lambda p^a+\alpha_1-\alpha_n)$; and contains restrictedly p^a, if $\lambda, \alpha_1-\alpha_2, \ldots \alpha_1-\alpha_n$ are each relatively prime to p; while, since $f(\beta)$ is rational, this must be an integral power of p. By application of the same principle that $f(x)$ is a (permissible mod. p) rational function of x, various restrictions upon the possible values of α may be obtained upon the supposition that $\alpha_1 \ldots \alpha_r$ is a set of cognate roots.

A striking restriction is that a *fractional* value of α is its *maximum* limit. For, if $\alpha_1+\mu p^{\alpha'}=$ integer $(\alpha'>\alpha=$ fraction$)$, then the difference $\mu p^{\alpha'}-\lambda p^{\alpha}=$ integer, which is therefore divisible by some power of p between α' and α; but this is impossible, since λ is relatively prime to p. Thus, for example, if the roots of $f(x)=0$ become integers by additions of fractional powers of a divisor, p, of its discriminant, $f(x)$ will not be rationally divisible by unlimited powers of p; for we have simply to take i great enough so that $i_1, \ldots i_n$ are greater than these fractional powers, in order to determine a power p^i which is not such a divisor.

It is not necessary, however, that α should take a fractional value in order to have a maximum; for, α being an integer, the variations of λ by integer additions give all integers of the form $\beta+np^{\alpha}$; and if we can determine the integer n so that $\lambda+n=M[p]$ and so increase α, λ pertains to an integral class mod. $[p]$. Therefore we can form from an integer β, and a non-integral quantity λ, mod. $[p]$, quantities $\alpha_1=\beta-\lambda p^{n_1}$ such that the integer n_1 is a maximum value of α.

The quantities λp^{α} which added to the roots of $f(x)=0$ give an integer β are given by the integral equation $f(\beta-\lambda p^{\alpha})=0$. On removing from this the factors p, the coefficients of powers of λ greater than the r^{th} must contain p,

since the $n-r$ solutions of λ corresponding to the roots outside the cognate set must be partial infinites (mod. p). Also similarly, as many of the coefficients of powers of λ which are $\overline{\gtrless} r$ will contain p as there are roots in the cognate set whose maximum powers of p for additions of which they become $=\beta$ are $<\alpha$. The remaining values of λ (mod. $[p]$) are obtained from the equation which results from dropping such coefficients; and according as a solution is or is not integral mod. $[p]$ can α be increased or not for a corresponding root of the cognate set. A particular case is $r=1$; when the resulting equation is linear, and has always integral solutions. In other words, if $f(x)$ is rationally divisible by p, corresponding to a root of $f(x)=0$ which has no cognates, mod. $[p]$, it is rationally divisible by unlimited powers of p, corresponding to that root.

In order that a quantity may become an integer by additions of some power of p, it is necessary and sufficient that it should satisfy $(P)=0$ mod. $[p]$. This modular equation may be replaced by an infinite series with respect to the modulus p, viz.: $$X^{\pi^i}-X^{\pi^{i-1}}=0 \text{ mod. } p \ldots (P_i).$$
Since we can fill the form for the residues of p^i in $\pi^{i-1}(\pi-1)$ ways so as to be relatively prime to p, this is the modular equation which is satisfied (mod. p^i) by every integral class;* but its discriminant, therefore, contains p. In fact, from the distributive property of the power π over a sum or difference (mod. p), $P_i \equiv P_1^{\pi^{i-1}}$ (mod. p). All solutions of (P_i) are therefore of the form $\beta+\lambda p^{\frac{1}{\pi^{i-1}}}$ where β is an integer; which includes, primitively, all quantities which can become integers by additions of p^α, α lying between $\frac{1}{\pi^{i-1}}$ and $\frac{1}{\pi^{i-2}}$. The solutions of (P_1) include the cases $\alpha \overline{\gtrless} 1$.

We may illustrate the preceding theory by the determination of the divisors of the cyclotomic functions. Take, for example, $f(x) \equiv x^4-x^3-x^2-2x+4$, whose roots are $\phi(\rho) \equiv \rho+\rho^4+\rho^{16}$, ρ being a primitive 21st root of unity. The powers of ρ for which $\phi(\rho)$ remains unchanged, which necessarily form a cycle mod. 21, are 1, 4, 16. If a prime p is a divisor, $\phi(\rho)$ must satisfy $x^p \equiv x$ mod. p; or since the power p is distributive mod. p, and the coefficients of ϕ integral,
$$\phi(\rho^p) \equiv \phi(\rho) \text{ mod. } p. \tag{1}$$
The first member becomes identically the second if $p \equiv 1, 4, 16$, mod. 21; which determines forms of primes which are divisors of $x^4-x^3-x^2-2x+4$, for

*The form of this modular equation (mod. p^i) shows that any set of incongruous classes (mod. p), raised to the π^{i-1} power, are the solutions of $(P)=0$ (mod. p^i). If in the latter so resolved into linear factors, we substitute $X^{\pi^{r-1}}$ for X, we obtain a resolution of (P_r) mod. p^i into its non-cognate modular factors.

integral or (permissible) fractional values of x. For a prime p which is relatively prime to 21, and not one of these forms ($\equiv 1, 4, 16$, mod. 21), the first member of (1) is some root of the cyclotomic different from the second member; and if the congruence subsists, p is a divisor of the discriminant of $f(x)$. Conversely, for a divisor p of the discriminant of $f(x)$, this congruence subsists mod. $[p]$, but not necessarily mod. p. To obtain the primes, fractional powers of which <1 are added to the roots to make them integers, we seek the primitive solutions of $x^{p^i} - x^{p^{i-1}} \equiv 0$ mod. p. The solutions can only be primitive when p is a divisor of the index 21. The value of $i-1$ is the power of p which is contained in the index; and $\rho^{p^{i-1}}$ is a primitive $\left(\frac{21}{p^{i-1}}\right)^{\text{st}}$ root of unity $= \rho_1$. Thus if $\phi(\rho_1^p) \equiv \phi(\rho_1)$ mod. p is satisfied, p is a divisor of $f(x)$. Whence either $p \equiv 1, 4, 16$, mod. $\frac{21}{p^{i-1}}$, or p is a divisor of the discriminant of the cyclotomic corresponding to $\phi(\rho_1)$. We have evidently $\phi(\rho) \equiv \phi(\rho_1)$ mod. p, so that the cyclotomic corresponding to the first member is a power $= h$ of the cyclotomic corresponding to the second; and its roots are made integral by additions of a multiple of p^{a}, where α is some quantity conditioned by the inequality. Thus no power of p greater than h^{a} $\frac{1}{h} \leqq \alpha < \frac{1}{p^{i-2}}$, can be a divisor of $f(x)$. For example, $f(x)$ is divisible by 7, but not by 7^2.

The problem of the determination of the divisors of a function may be generalized, if instead of seeking the moduli for which a given function has a rational linear factor, we seek the orders of the rational modular divisors of a function with respect to a given modulus p^i. This more general problem we shall consider hereafter.

An important property of the π^{th} power is that it is distributive (mod. p) over a sum or difference. For, A and X being integral classes, so is $A + X$, so that $(A+X)^\pi = A + X = A^\pi + X^\pi$; therefore, the modular equation of $(\pi-1)^{\text{st}}$ degree, $(A+X)^\pi = A^\pi + X^\pi$ is *identical,* since it has π solutions (the integral classes) whose differences are relatively prime to the modulus.

This result leads to the conclusion that any system of integers such that the residues of a finite integer are finite in number consists of a factoring of ordinary integers;* for any prime p of such a system is contained in the ordinary binomial coefficients π, $\frac{\pi(\pi-1)}{1.2}$, . . . , and it readily follows that any integer is contained in the ordinary integer which represents the number of its residues. The number

* Ordinary integers are included in any system in the sense of representing repetitions of the unit of the system.

of residues of an integer may be called its *norm;* from which definition follows the property that the norm of a product is the product of the norms of the factors. We shall call the *least* ordinary integer which contains a given integer, its *subnorm;* from which definition follow the properties: The subnorm of a prime is an ordinary prime; of a power of a prime, an ordinary prime or a power of it; of a product, the least common multiple of the subnorms of the relatively prime factors. An ordinary prime is the subnorm of any of its factors. Those residues of a modulus which can be expressed as ordinary integers are precisely the ordinary residues of its subnorm. If a power n is distributive with respect to a modulus, the binomial coefficients $n, \frac{n.n-1}{1.2}, \ldots$ must contain the subnorm of the modulus. Hence, if we assume it as a property of these binomial coefficients, that they can have no ordinary common factor except a prime when n is a power of that prime, then, if, and only if the modulus is a factor of an ordinary prime will there exist distributive powers with respect to it, viz., the ordinary prime and its powers only. This theorem may be otherwise obtained as follows:

Firstly, n being a distributive power, it contains the modulus. The immediate consequence is that the discriminant $(n-1)^{n-1}$ of $X^n - X$ is relatively prime to the modulus. Secondly, n being a distributive power, the solutions of $X^n - X = 0$ repeat themselves by addition—*e. g.*, $(A+B)^n = A^n + B^n = A + B$. These conditions are sufficient, since we then have, with respect to each relatively prime factor (p^i) of the modulus as modulus, the *identity* $(1+X)^n = 1 + X^n$, *i. e.*, it is satisfied by the n solutions of $X^n - X = 0$ whose differences are relatively prime to the modulus. Now, any ordinary residue of the subnorm of p^i is a solution (mod. p^i), viz., $1 + 1 + \ldots = \lambda$; whence the subnorm must be an ordinary prime ν, since otherwise differences of the residues of ν would contain the prime factor of ν, and therefore the prime p. Again, each solution (mod. p^i) excepting zero, is relatively prime to p^i, and therefore its repetition gives rise to as many different solutions as there are ordinary integer residues of the subnorm ν; and if $1, X_2, \ldots X_s$ be a complete set of linearly independent solutions, any solution may be thrown in only one way into the form $\lambda_1 + \lambda_2 X_2 + \ldots \lambda_s X_s$, where $\lambda_1, \ldots \lambda_s$ are ordinary residues of ν. The whole number of solutions is therefore $\nu^s = n$; an equality which can only be possible for each relatively prime factor of the modulus when all have the same subnorm ν. Finally, if the subnorm of the modulus is an ordinary prime ν, then ν itself satisfies the two conditions of a distributive power; and as a consequence ν^s is a distributive power.

As we have seen, the norm of a prime modulus is a distributive power, and it is therefore a power of the subnorm. The preceding demonstration also shows, if $\nu^f = \pi$, that f represents the number of linearly independent residues of p. We shall use the term *degree*, in general, to signify the number of linearly independent residues. Depending upon the distributive property of the subnorm ν, is the proposition that if the degree of p is > 1 there exist no primitive integral solutions of $X^{\pi^i} - X^{\pi^{i-1}} = 0$ mod. p^i. For, α being an integer, we have successively, $\alpha^\pi = \alpha + Mp$, $\alpha^{\pi\nu} = \alpha^\nu + Mp^2, \ldots \alpha^{\pi\nu^{i-1}} = \alpha^{\nu^{i-1}} + Mp^i$, so that every integral class satisfies a modular equation $X^{\pi\nu^{i-1}} - X^{\nu^{i-1}} = 0$ (mod. p^i) of less degree than that of the first modular equation, if $\pi > \nu$. Primitive solutions of this latter equation always exist in the special case $i = 2$, viz., $\alpha + \beta p$, β being relatively prime to p and α a primitive solution (including zero) for $i = 1$. If ν does not contain a power of p, primitive solutions always exist, namely, those for p^2. If, however, the *multiplicity* of p in ν is > 1, every integral class satisfies ($i > 2$) a modular equation of this same form, of degree $\pi\nu^s$, $s < i - 1$, for which primitive solutions exist. In the case $\pi = \nu$, p being contained but once in ν, the residues of p^i are ordinary integers, and the primitive solutions are identical with those for ν^i. The number of primitive solutions is in the several cases $\pi^{i-2}(\pi - 1)\{\tau(\pi - 1) + 1\}$. The symbol τ coincides with the "totient of" except for $\pi = 2^f$, when there are no primitive solutions which are relatively prime to p unless the degree of the corresponding modular equation be still further reduced.

[TO BE CONTINUED.]

Proof of a Theorem in Partitions.

By Morgan Jenkins, *London.*

In the *American Journal of Mathematics*, Vol. V, No. 3, p. 257, Professor Sylvester states that the following theorem, although confirmed by a multitude of instances, remains to be proved.

If a square matrix, M_1, of the j^{th} order be formed in which the elements of a diagonal (say of the diagonal which descends from left to right) are all $i+1$, the elements below the diagonal all unity, and those above the diagonal all zeros; and if all possible sets of q rows of this matrix be added together, and the series so formed be added to all possible regularized partitions of content less than n so as to produce a partition, not necessarily regularized, of content n; then the number of times such partition of content n will appear, variously arranged, in P_q, that is in the table formed as above by the addition of the sums of q rows, will be ${}_\mu C_q$ or $\dfrac{\mu(\mu-1)(\mu-2)\ldots(\mu-q+1)}{1.2.3\ldots q}$ ways, where μ is the number of different elements in the given partition of content n which are greater than i. The proof here offered depends on the consideration of the nature of the series formed from the matrix M_1, and on reversing the order of construction the series obtained from M_1 are of the form

$qqq\ldots(i+q)q-1,\ q-1\ldots(i+q-1)\ldots(i+k+1),\ k,\ k,\ldots(i+k)\ldots(i+1)\,0,\ 0.$

The elements $i+q,\ i+q-1,\ldots i+1$ must be in descending order of magnitude; but there is nothing to prevent q being greater than i; the number of elements in, or the extent of the series is j; the elements, if any, between $i+k+1$ and $i+k$ must be $=k$; but there may be none such: so the elements, if any, to the left of $i+q$ must be q, and the zeros, if any, must be to the right of $i+1$.

Now take a given partition of content n and extent j; if we select q elements greater than i, but containing repetitions, no regularized partition cor-

responding to these q elements can be obtained, because on subtracting the sequence $i+q,\ i+q-1, \ldots i+1$ from these q elements the remainder would not be regularized; but if the given partition contain μ different elements greater than i, and we select any q of these and ask what must be the mode or modes of arranging the elements of the given partition in order that, after subtracting from it a q-series, derived as above (the elements $i+q,\ i+q-1, \ldots i+1$ being subtracted from the selected q elements) the remainder may be a regularized partition of content less than n, the answer is there is one and only one mode of so arranging the given partition: also there is only one possible subtrahend and one regularized partition of content less than n corresponding to these q elements.

An example will show the mode of arrangement and will lead to the proof. Let 16, 14, 13, 12, 11, 11, 9, 8, 8, 6, 5, 5, 4, 3, 3, 2, 0, 0 be the given partition, n being $=130$, $j=18$; let $i=3$, then $\mu=10$; let $q=5$, and the five selected elements be 12, 11, 9, 8, 5; then the elements $i+q,\ i+q-1, \ldots i+1$ are 8, 7, 6, 5, 4; and they must be arranged in descending order; for otherwise the remainders would be unregularized, that is, not in descending order of magnitude.

Again write in descending order the remaining elements of the minuend, viz. 16, 14, 13, 11, | 8, | 6, | 5, | 4, 3, 3, 2, | 0, 0; and it will be found that they must be divided as marked: the 0, 0 must be placed to the right of the 5 in the whole minuend, the 4, 3, 3, 2 between the 8 and the 5; and so on, as indicated below

$$\begin{array}{cccccccccccccccccc}
16, & 14, & 13, & 11, & \overset{*}{12}, & 8, & \overset{*}{11}, & 6, & \overset{*}{9}, & 5, & \overset{*}{8}, & 4, & 3, & 3, & 2, & \overset{*}{5}, & 0, & 0 \\
5, & 5, & 5, & 5, & 8, & 4, & 7, & 3, & 6, & 2, & 5, & 1, & 1, & 1, & 1, & 4, & 0, & 0 \\
\hline
11, & 9, & 8, & 6, & \underset{*}{4}, & 4, & \underset{*}{4}, & 3, & \underset{*}{3}, & 3, & \underset{*}{3}, & 3, & 2, & 2, & 1, & \underset{*}{1}, & 0, & 0
\end{array}$$

The 5 selected elements in the minuend and the corresponding remainders are marked with a star: the 2 in the minuend cannot be placed to the right of the $\overset{*}{5}$; for then on subtracting 0 from 2 the remainder 2 would come to the right of $\underset{*}{1}$, producing an irregularity: 4 in the minuend must come to the right of $\overset{*}{8}$; for if it came to the left 2 or more from 4 would leave 2 or less coming to the left of $\underset{*}{3}$, producing an irregularity: thus it may be shown that 4, 3, 3, 2 must come between $\overset{*}{8}$ and $\overset{*}{5}$, and, if so, then in descending order, because, when equal 1's are subtracted, the remainders are to be regularized; and so on for the remaining sets.

Similarly in other cases after subtracting $i+q$, $i+q-1, \ldots i+1$, respectively, from the q selected elements arranged in descending order, we must arrange the remaining $j-q$ elements in descending order, then divide them into $q+1$ sets (some sets possibly containing no elements) and place the $q+1^{\text{th}}$ set in descending order to the right of $i+1$, the q^{th} set between $i+2$ and $i+1$, and so on; the 1st set from the left being to the left of $i+q$.

If Q_{k+1} and Q_k are the elements from which $i+k+1$ and $i+k$ respectively, are subtracted, then any other element m of the $j-q$ elements will be placed between Q_{k+1} and Q_k if $m \overline{\overline{<}}\, Q_{k+1}-(i+1)$ and $\overline{\overline{>}}\, Q_k-(i+k)$, that is if m is between $Q_{k+1}-i-1$ and Q_k-i or equal to either of them: any element which is $< Q_1-i$ going to the right of Q_1 and any element which is $> Q_q-i-1$ going to the left of Q_q.

The desired result now follows; for to every selection of q different elements there is one and only one arrangement of the partition of content n appearing in the table P_q: also there is no arrangement of the given partition corresponding to a selection of q elements containing repetitions, that is, in which $i+q$, $i+q-1, \ldots i+1$ are subtracted from those elements. Hence the given partition will appear ${}_\mu C_q$ times in the table P_q.

Further List of Corrections suggested by M. Jenkins to Prof. Sylvester's Constructive Theory of Partitions.

American Journal of Mathematics, Vol. V, Nos. 3 and 4.

Vol. V, No. 3, p. 255, 8 lines from end, $2n-(i+3)$ should be $n-(i+3)$.

Page 256, between 2d and 3d rows of sinister table insert 13.2.0

" " " 7th and 8th " " " " 11.2.2

" " in 6th row of dexter table, for 8.4.3(2) write 8.4.3(1).

" 261, line 11 from the end, interchange protraction and contraction so as to read "*contraction* could not now be applied to A' and B' nor *protraction* to C'."

" 263, line 21. If $f(x)=(1-x)(1-x^3)(1-x^3)(1-x^7)(1-x^9)$, for the second x^3 read x^5.

" " line 25, for 'latter' read 'former'.

" 265, line 29, for l^r read l^λ.

" 270, line 11, for $1+2$ read $i+2$.

" " line 12, for $1+2$ read $i+2$.

" 272, line 7, for $X_j x^{\frac{i^2+i}{2}}$ read $X_j x^{\frac{j^2+j}{2}}$.

" " line 14, for 'the minimum negative residue of $i-1$' read $i+1$.

" 274, line 6 from end, for $\dfrac{x^{\frac{1}{2}n(n+1)}}{1-x^n}$ read $\dfrac{x^{\frac{1}{2}r(r+1)}}{1-x^r}$.

" 275, line 9, for 'to the 5th now' read 'to the 5th row now.'

" 276, line 21, for 15, 7, 3 read 13, 11, 3.

" " line 22, for $(1+ax)(1-ax^3)(1-ax^j)\ldots$ read $(1+ax)(1+ax^3)\ldots(1+ax^{2j-1})$.

" " line 24, for $\dfrac{x}{1-x}\alpha$ read $\dfrac{x}{1-x^2}\alpha$.

" " line 25, for 'angle whose *nodes* contain i nodes' read whose *sides*.

Page 277, line 9, for 'with $j-i$ or fewer parts' read $j-1$.

" " line 14, for $1+\dfrac{1-x^{\omega+1}}{1-x^2}x^{\omega}+\dfrac{1-x^{\omega+1}.1-x^{\omega+3}}{1-x.1-x^4}x^{\omega+1}$ etc.

$$\text{read } x^{\omega}+\frac{1-x^{\omega-1}}{1-x^2}x^{\omega+1}+\frac{1-x^{\omega-1}.1-x^{\omega-3}}{1-x^2.1-x^4}x^{\omega+4}+\text{etc.}$$

If in the expression in line 11, viz. in

$$\frac{1-x^{2i-2j+2}.1-x^{2i-2j+4}\ldots 1-x^{2i-2}}{1-x^2.1-x^4\ldots 1-x^{2j-2}}x^{j^2-2j+2i}, \text{ we put } j=3 \text{ we obtain}$$

$$\frac{1-x^{2i-4}.1-x^{2i-2}}{1-x^2.1-x^4}.x^{9-6+2i}=\frac{1-x^{2i-2}.1-x^{2i-4}}{1-x^2.1-x^4}.x^{2i+3}=\frac{1-x^{\omega-1}.1-x^{\omega-3}}{1-x^2.1-x^4}.x^{\omega+4},$$

since $\omega=2i-1$, and similarly for other terms when we put $j=2$ and $j=1$.

The correction which I offer seems to me to be right, and the expression in the paper to give a wrong result in the case when n happens to be equal to $\omega+2$: for then the number of parts being supposed to be exactly i, the first bend contains $2i-1$ or ω nodes, and there is then no way of placing the remaining 2 nodes so as to make the partition a conjugate partition—supposing I have not misunderstood the article.

Page 278, line 13, for 19, 7, 6, 6 read 10, 7, 6, 6.

" 279, figure, either insert a node at junction of 5th column and 7th row or remove a node from junction of 7th column and 5th row.

" " lines 6 and 7, if we remove a node from the figure no change is required in these two lines; but if we insert a node in the figure, then 11 11 11 7 3 3 should be 11 11 11 7 5 3 and 5 5 5 3 1 1 should be 5 5 5 3 2 1.

" 280, line 5 from end, after $\dfrac{1}{1-ax.1-ax^2\ldots 1-ax^{\theta}}$ insert 'or of $x^n a^j$.'

" 283, line 3, for a^j read a^{θ}.

" " line 4, for $(x^{\theta}+ax^{1\theta})\}$ read $(x^{\theta}+x^{2\theta})$.

" 285, line 1, for $\dfrac{l_1(2-j-1)}{2}$ read $\dfrac{l_1-(2j-1)}{2}$.

" " line 6 from end omitting notes, for x^n read $x^{\frac{n}{2}}$.

" " line 7, for x^{2i+1} read x^{2i+2}.

" 288, a_i-i is, I believe, the right final term; but it appears as if it were the first of a pair instead of the last of a pair, a_i-i being a quantity which may vanish.

If the pair of expressions which in the text precede $a_i - i$, if definitely expressed and not left to be understood, should be

$$[a_{i-1} + \alpha_{i-1} - (2i - 3)],\ [a_{i-1} + \alpha_{i-2} - (2i - 2)],$$

and not as in the text $[a_{i-1} + \alpha_{i-1} - (2i - 1)]$, $[a_{i-1} + \alpha_i - 2i]$, the factor which should precede $a_i - i$ is $[a_i + \alpha_i - (2i - 1)]$.

I do not quite follow the first 5 lines of p. 289 (in No. 4), possibly from the oversight in the subscripts I do not see what is intended. But in seems to me the following proof would be right:

The expressions of the same form succeeding $a_1 + \alpha_1 - 1$ and $a_1 + \alpha_2 - 2$ must be continued so long as they are positive, and must be rejected when they become negative.

Now from the fact of i being the content of the side of the square belonging to the transverse graph $a_i =$ or $> i$, $\alpha_i =$ or $> i$, therefore $a_i + \alpha_i - (2i - 1)$ is positive and is therefore one of the terms of the series. Also $a_{i+1} =$ or $< i$ and $\alpha_{i+1} =$ or $< i$, therefore $a_{i+1} + \alpha_{i+1} - (2i + 1)$ is negative and must consequently be rejected.

The intermediate expression is $a_i + \alpha_{i+1} - 2i$; and for this we may in all cases put $a_i - i$ as the last term of the series for the following reason:

If the extreme inside bend have more than one node in the row, then $\alpha_{i+1} = i$ and $a_i + \alpha_{i+1} - 2i$ is $= a_i - i$, which is not negative since $a_i =$ or $> i$. If the extreme inside bend degenerate, so that it consists only of a vertical line or of a single point, then $a_i = i$; and since $\alpha_{i+1} < i$ in this case, therefore $a_i + \alpha_{i+1} - 2i$ is negative and inadmissible as a term in the series; but since $a_i - i = 0$ there is no harm in putting it as the final term in the series.

VOL. V, NO. 2, ON SUBINVARIANTS, *i. e.* SEMI-INVARIANTS TO BINARY QUANTICS OF AN UNLIMITED ORDER.

Page 114, last line, for 3100 read 3110.

On Theta-Functions with Complex Characteristics.

By Thomas Craig, *Johns Hopkins University.*

Prym in his memoir "Untersuchungen über die Riemann'sche Thetaformel und die Riemann'sche Charakteristikentheorie" considers characteristics

$$\begin{pmatrix} g_1 \, g_2 \ldots g_p \\ h_1 \, h_2 \ldots h_p \end{pmatrix}$$

where the quantities g, h are any constants instead of being as usual integers. If the quantities g and h are rational fractions with a common denominator, say r, then it is shown that there are in all r^{2p} different functions. Krazer in his Habilitationsschrift considers the problem of finding the relations connecting these r^{2p} different theta-functions, limiting himself however to the case of $p = 1$, $r = 3$. In what follows I have just begun a study of the theta-functions corresponding to the case of a characteristic made up of complex quantities. I have called the functions Ξ-functions to distinguish them from the ordinary theta-functions when the characteristic is made up of integers.

The theta-function of p variables is defined by the equation

$$\vartheta\begin{pmatrix} l_1, & l_2 & \ldots & l_p \\ \lambda_1, & \lambda_2 & \ldots & \lambda_p \end{pmatrix}(v_1, \; v_2 \ldots v_p) = \Sigma \text{ exp.} \begin{pmatrix} n_1 + l_1, & n_2 + l_2 & \ldots & n_p + l_p \\ v_1 + \lambda_1, & v_2 + \lambda_2 & \ldots & v_p + \lambda_p \end{pmatrix},$$

where

$$\begin{pmatrix} n_1 + l_1, & n_2 + l_2 & \ldots & n_p + l_p \\ v_1 + \lambda_1, & v_2 + \lambda_2 & \ldots & v_p + \lambda_p \end{pmatrix} = \tfrac{1}{4}(*\!\!\not\,)(n_1 + l_1, \; n_2 + l_2 \ldots n_p + l_p)^2$$
$$+ \tfrac{1}{2}\pi i \{(n_1 + l_1)(v_1 + \lambda_1) + \ldots + (n_p + l_p)(v_p + \lambda_p)\}$$

and $(*)$ denotes the constants

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \ldots & a_{1p} \\ & a_{22} & a_{23} & \ldots & a_{2p} \\ & & a_{33} & \ldots & a_{3p} \\ & & \ldots & \ldots & \ldots \\ & & & & a_{pp}. \end{array}$$

It will be understood of course that $a_{rs} = a_{sr}$.

In the case of the theta-functions the characteristic

$$\begin{pmatrix} l_1, & l_2 & \ldots & l_p \\ \lambda_1, & \lambda_2 & \ldots & \lambda_p \end{pmatrix}$$

may be made up of positive or negative integers, but, as a matter of fact, l and λ are always made equal to either zero or unity. In what follows I propose to examine the functions for which l, λ are complex quantities, say

$$l_r = a_r + ib_r$$
$$\lambda_r = \alpha_r + i\beta_r,$$

and particularly the cases when a, b, α, β take the values zero and unity, and also when

$$a_r + ib_r \text{ and } \alpha_r + i\beta_r$$

are conjugate complex quantities. I will denote the functions under consideration by the symbol Ξ, that is

$$1.\quad \Xi\begin{pmatrix} l_1, & l_2 \ldots l_p \\ \lambda_1, & \lambda_2 \ldots \lambda_p \end{pmatrix}(v_1,\ v_2 \ldots v_p) = \Sigma \text{ exp.} \begin{pmatrix} n_1 + l_1, & n_2 + l_2 \ldots n_p + l_p \\ v_1 + \lambda_1, & v_2 + \lambda_2 \ldots v_p + \lambda_p \end{pmatrix}.$$

It will be understood in all that follows, when the symbol Ξ is used, that the letters l and λ denote complex quantities, and when ϑ is used that they denote positive or negative integers, or, as usual, simply zero and unity. The convergency of the Ξ-functions is secured as in the case of the ϑ-functions: "the parameters a_{rs} may be real or imaginary, but they must be such that reducing each of them to its real part the resulting function $(*\between n_1, n_2 \ldots n_p)^2$ is invariable in its sign, and negative for all real values of n_r".* I will assume at once that a_r, b_r, α_r, $\beta_r = 0$ or 1; the total number of Ξ-functions is now easily found. Denote by (a, α) the entire set of a's and α's, and by (b, β) the entire set of b's and β's. Assume first that $(b, \beta) = 0$ and give (a, α) all possible values made up of 0 and 1; we have then 2^{2p}, or 4^p functions. Now give (b, β) all possible values made up of zero and unity, while $(a, \alpha) = 0$, and we have again 4^p functions, so that finally we have as the total number of Ξ-functions $(4^p)^2$ or 4^{2p}. Thus the total number of p-tuple Ξ-functions is equal to the square of the total number of ϑ-functions. For example:

	Total number of ϑ-functions.	Total number of Ξ-functions.
$p = 1$	4	16
$p = 2$	16	256
$p = 3$	64	4096
$p = 4$	256	65536
etc.	etc.	etc.
. . .	. . .	. . .
$p = p$	2^{2p}	2^{4p}.

* A Memoir on the Single and Double Theta-Functions, by A. Cayley (page 899), *Phil. Trans.*, 1880.

For the case of $p = 1$ the Ξ-characteristics are as follows:

$$
\begin{matrix}
\binom{0}{0} & \binom{i}{0} & \binom{0}{i} & \binom{i}{i} \\
\binom{1}{0} & \binom{1+i}{0} & \binom{1}{i} & \binom{1+i}{i} \\
\binom{0}{1} & \binom{i}{1} & \binom{0}{1+i} & \binom{i}{1+i} \\
\binom{1}{1} & \binom{1+i}{1} & \binom{1}{1+i} & \binom{1+i}{1+i}
\end{matrix} \qquad (2.)
$$

In order to form the characteristics for the case of $p = 2$ we have only to combine the following two tables, the first of which corresponds to $(b, \beta) = 0$, and the second to $(a, \alpha) = 0$,

$$
\begin{array}{|cccc|}
\hline
00 & 10 & 01 & 11 \\
00 & 00 & 00 & 00 \\
\hline
00 & 10 & 01 & 11 \\
10 & 10 & 10 & 10 \\
\hline
00 & 10 & 01 & 11 \\
01 & 01 & 01 & 01 \\
\hline
00 & 10 & 01 & 11 \\
11 & 11 & 11 & 11 \\
\hline
\end{array}
\qquad
\begin{array}{|cccc|}
\hline
0\,0 & i\,0 & 0\,i & i\,i \\
0\,0 & 0\,0 & 0\,0 & 0\,0 \\
\hline
0\,0 & i\,0 & 0\,i & i\,i \\
i\,0 & i\,0 & i\,0 & i\,0 \\
\hline
0\,0 & i\,0 & 0\,i & i\,i \\
0\,i & 0\,i & 0\,i & 0\,i \\
\hline
0\,0 & i\,0 & 0\,i & i\,i \\
i\,i & i\,i & i\,i & i\,i \\
\hline
\end{array} \qquad (3.)
$$

We have then for the Ξ characteristics in the case of $p = 2$ the following table:

4.

0 0 0 0	i 0 0 0	0 i 0 0	i i 0 0	0 0 i 0	i 0 i 0	0 i i 0	i i i 0	0 0 0 i	i 0 0 i	0 i 0 i	i i 0 i	0 0 i i	i 0 i i	0 i i i	i i i i
1 0 0 0	1+i 0 0 0	1 i 0 0	1+i i 0 0	1 0 i 0	1+i 0 i 0	1 i i 0	1+i i i 0	1 0 0 i	1+i 0 0 i	1 i 0 i	1+i i 0 i	1 0 i i	1+i 0 i i	1 i i i	1+i i i i
0 1 0 0	i 1 0 0	0 1+i 0 0	i 1+i 0 0	0 1 i 0	i 1 i 0	0 1+i i 0	i 1+i i 0	0 1 0 i	i 1 0 i	0 1+i 0 i	i 1+i 0 i	0 1 i i	i 1 i i	0 1+i i i	i 1+i i i
1 1 0 0	1+i 1 0 0	1 1+i 0 0	1+i 1+i 0 0	1 1 i 0	1+i 1 i 0	1 1+i i 0	1+i 1+i i 0	1 1 0 i	1+i 1 0 i	1 1+i 0 i	1+i 1+i 0 i	1 1 i i	1+i 1 i i	1 1+i i i	1+i 1+i i i
0 0 1 0	i 0 1 0	0 i 1 0	i i 1 0	0 0 1+i 0	i 0 1+i 0	0 i 1+i 0	i i 1+i 0	0 0 1 i	i 0 1 i	0 i 1 i	i i 1 i	0 0 1+i i	i 0 1+i i	0 i 1+i i	i i 1+i i
1 0 1 0	1+i 0 1 0	1 i 1 0	1+i i 1 0	1 0 1+i 0	1+i 0 1+i 0	1 i 1+i 0	1+i i 1+i 0	1 0 1 i	1+i 0 1 i	1 i 1 i	1+i i 1 i	1 0 1+i i	1+i 0 1+i i	1 i 1+i i	1+i i 1+i i
0 1 1 0	i 1 1 0	0 1+i 1 0	i 1+i 1 0	0 1 1+i 0	i 1 1+i 0	0 1+i 1+i 0	i 1+i 1+i 0	0 1 1 i	i 1 1 i	0 1+i 1 i	i 1+i 1 i	0 1 1+i i	i 1 1+i i	0 1+i 1+i i	i 1+i 1+i i
1 1 1 0	1+i 1 1 0	1 1+i 1 0	1+i 1+i 1 0	1 1 1+i 0	1+i 1 1+i 0	1 1+i 1+i 0	1+i 1+i 1+i 0	1 1 1 i	1+i 1 1 i	1 1+i 1 i	1+i 1+i 1 i	1 1 1+i i	1+i 1 1+i i	1 1+i 1+i i	1+i 1+i 1+i i
0 0 0 1	i 0 0 1	0 i 0 1	i i 0 1	0 0 i 1	i 0 i 1	0 i i 1	i i i 1	0 0 0 1+i	i 0 0 1+i	0 i 0 1+i	i i 0 1+i	0 0 i 1+i	i 0 i 1+i	0 i i 1+i	i 0 i 1+i
1 0 0 1	1+i 0 0 1	1 i 0 1	1+i i 0 1	1 0 i 1	1+i 0 i 1	1 i i 1	1+i i i 1	1 0 0 1+i	1+i 0 0 1+i	1 i 0 1+i	1+i i 0 1+i	1 0 i 1+i	1+i 0 i 1+i	1 i i 1+i	1+i i i 1+i
0 1 0 1	i 1 0 1	0 1+i 0 1	i 1+i 0 1	0 1 i 1	i 1 i 1	0 1+i i 1	i 1+i i 1	0 1 0 1+i	i 1 0 1+i	0 1+i 0 1+i	i 1+i 0 1+i	0 1 i 1+i	i 1 i 1+i	0 1+i i 1+i	i 1+i i 1+i
1 1 0 1	1+i 1 0 1	1 1+i 0 1	1+i 1+i 0 1	1 1 i 1	1+i 1 i 1	1 1+i i 1	1+i 1+i i 1	1 1 0 1+i	1+i 1 0 1+i	1 1+i 0 1+i	1+i 1+i 0 1+i	1 1 i 1+i	1+i 1 i 1+i	1 1+i i 1+i	1+i 1+i i 1+i
0 0 1 1	i 0 1 1	0 i 1 1	i i 1 1	0 0 1+i 1	i 0 1+i 1	0 i 1+i 1	i i 1+i 1	0 0 1 1+i	i 0 1 1+i	0 i 1 1+i	i i 1 1+i	0 0 1+i 1+i	i 0 1+i 1+i	0 i 1+i 1+i	i i 1+i 1+i
1 0 1 1	1+i 0 1 1	1 i 1 1	1+i i 1 1	1 0 1+i 1	1+i 0 1+i 1	1 i 1+i 1	1+i i 1+i 1	1 0 1 1+i	1+i 0 1 1+i	1 i 1 1+i	1+i i 1 1+i	1 0 1+i 1+i	1+i 0 1+i 1+i	1 i 1+i 1+i	1+i i 1+i 1+i
0 1 1 1	i 1 1 1	0 1+i 1 1	i 1+i 1 1	0 1 1+i 1	i 1 1+i 1	0 1+i 1+i 1	i 1+i 1+i 1	0 1 1 1+i	i 1 1 1+i	0 1+i 1 1+i	i 1+i 1 1+i	0 1 1+i 1+i	i 1 1+i 1+i	0 1+i 1+i 1+i	i 1+i 1+i 1+i
1 1 1 1	1+i 1 1 1	1 1+i 1 1	1+i 1+i 1 1	1 1 1+i 1	1+i 1 1+i 1	1 1+i 1+i 1	1+i 1+i 1+i 1	1 1 1 1+i	1+i 1 1 1+i	1 1+i 1 1+i	1+i 1+i 1 1+i	1 1 1+i 1+i	1+i 1 1+i 1+i	1 1+i 1+i 1+i	1+i 1+i 1+i 1+i

If the elements of the characteristics are conjugate imaginaries, that is, if we have $\alpha_r + i\beta_r = a_r - ib_r$, the total number of Ξ-functions for any given number of arguments is equal to the total number of theta-functions for the same number of arguments, *i. e.* in the case of p-tuple functions if the characteristic is

$$\begin{pmatrix} a_1 + ib_1, & a_2 + ib_2 \dots & a_p + ib_p \\ a_1 - ib_1, & a_2 - ib_2 \dots & a_p - ib_p \end{pmatrix}$$

(where a, b take the values zero and unity), we have 4^p Ξ-functions. For $p = 1$ the characteristics are

5. $$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ -i \end{pmatrix}, \begin{pmatrix} 1+i \\ 1-i \end{pmatrix};$$

for $p = 2$ they are

6. $$\begin{gathered} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1+i \\ 0 & 1-i \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \begin{pmatrix} 1 & 1+i \\ 1 & 1-i \end{pmatrix} \\ \begin{pmatrix} i & 0 \\ -i & 0 \end{pmatrix}, \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}, \begin{pmatrix} i & i \\ -i & -i \end{pmatrix}, \begin{pmatrix} i & 1+i \\ -i & 1-i \end{pmatrix} \\ \begin{pmatrix} 1+i & 0 \\ 1-i & 0 \end{pmatrix}, \begin{pmatrix} 1+i & 1 \\ 1-i & 1 \end{pmatrix}, \begin{pmatrix} 1+i & i \\ 1-i & -i \end{pmatrix}, \begin{pmatrix} 1+i & 1+i \\ 1-i & 1-i \end{pmatrix}. \end{gathered}$$

If in the function

$$\Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r), \quad r = 1, 2, \dots p,$$

we change l_r into $l_r + 2y_r$, where y_r is an integer, there will obviously be no change in the value of the function. If, however, we change λ_r into $\lambda_r + 2x_r$, where x_r is again an integer, we shall have as the new linear term in the exponential $\frac{1}{2}\pi i\{(n_1 + l_1)(v_1 + \lambda_1 + 2x_1) + \dots + (n_p + l_p)(v_p + \lambda_p + 2x_p)\}$, which differs from the original term by the quantity

$$\pi i \Sigma n_r x_r + \pi i \Sigma l_r x_r, \quad r = 1, 2, \dots p.$$

The term $\pi i \Sigma n_r x_r$ is, since n is even, an even multiple of πi and so produces no change in the value of the function; the second term $\pi i \Sigma l_r x_r$ causes the function to be multiplied by the factor $(-)^{\Sigma l_r x_r}$, that is, we have

7. $$\Xi\begin{pmatrix} l_r + 2y_r \\ \lambda_r + 2x_r \end{pmatrix}(v_r) = (-)^{\Sigma l_r x_r}\, \Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r);$$

since $l_r = a_r + ib_r =$, say, $A_r e^{i\phi_r}$, this may be written

8. $$\Xi\begin{pmatrix} l_r + 2y_r \\ \lambda_r + 2x_r \end{pmatrix}(v_r) = (-)^{\Sigma x_r A_r e^{i\phi_r}} . \Xi\begin{pmatrix} l_r \\ \lambda_r \end{pmatrix}(v_r).$$

On the assumption that a and b are each either zero or unity the modulus of l is either 0, 1, or $\sqrt{2}$, so that in the sum $\Sigma x_r A_r e^{i\phi_r}$ each term is either 0, $xe^{i\phi}$ or $x\sqrt{2}e^{i\phi}$, and therefore the factor $(-)^{\Sigma x_r A_r e^{i\phi_r}}$ in the preceding equation takes in all 2^{2p} forms; these are not all distinct and only one of them makes the factor

$=1$, viz. when all the moduli A are $=0$. Since the four possible moduli of l are $A=0,\ 1,\ 1,\ \sqrt{2}$, it would appear that the total number of independent values taken by $\Sigma x_r A_r e^{i\phi_r}$ is equal to 3^p. Among these 3^p quantities a certain number would seem to be repeated two or more (always an even number) of times to make up the 2^{2p} values of $\Sigma x_r A_r e^{i\phi_r}$. It is easy to see that the total number of values of this quantity which do not seem to be repeated is equal to $3^{p-2}.2^2$ (for $p \gtreqless 2$), and so the total number of values which are repeated is

$$3^p - 3^{p-2}.2^2 = 3^{p-2}.(3^2 - 2^2) = 5.3^{p-2}.$$

The moduli having the above values it is easy to see what the corresponding arguments are, and so to show that the above conclusions do not hold. Take the element l, $= a + ib$. This may also be written $l = Ae^{i\phi}$; suppose now that $a=0$ and $b=0$, then $A=0$ and ϕ is indeterminate, but since in this case the term $xAe^{i\phi}$ vanishes, it is unnecessary to consider ϕ at all. Suppose next that $b=0$ and $a=1$, then we have at once $\phi=0$, or 2π; now let $a=0$ and $b=1$ and we have $\phi=\frac{\pi}{2}$; finally if $a=1$ and $b=1$ we find $\phi=\frac{\pi}{4}$.

Take as illustration the case of the double Ξ-functions where the characteristic is

$$\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix} \equiv \begin{pmatrix} a_1 + ib_1 & a_2 + ib_2 \\ \alpha_1 + i\beta_1 & \alpha_2 + i\beta_2 \end{pmatrix};$$

the quantity to be examined is

$$(-)^{x_1 A_1 e^{i\phi_1} + x_2 A_2 e^{i\phi_2}}, \text{ say } (-)^R;$$

now R takes 16 values, these are

$\left\{\begin{matrix} a_1=0,\ b_1=0 \\ a_2=0,\ b_2=0 \end{matrix}\right\} 0$	$\left\{\begin{matrix} a_1=0,\ b_1=0 \\ a_2=1,\ b_2=0 \end{matrix}\right\} x_2.1.e^{i0}$
$\left\{\begin{matrix} a_1=0,\ b_1=0 \\ a_2=0,\ b_2=1 \end{matrix}\right\} x_2.1.e^{i\frac{\pi}{2}}$	$\left\{\begin{matrix} a_1=0,\ b_1=0 \\ a_2=1,\ b_2=1 \end{matrix}\right\} x_2\sqrt{2}.e^{i\frac{\pi}{4}}$
$\left\{\begin{matrix} a_1=1,\ b_1=0 \\ a_2=0,\ b_2=0 \end{matrix}\right\} x_1.1.e^{i0}$	$\left\{\begin{matrix} a_1=1,\ b_1=0 \\ a_2=1,\ b_2=0 \end{matrix}\right\} x_1.1.e^{i0} + x_2.1.e^{i0}$
$\left\{\begin{matrix} a_1=1,\ b_1=0 \\ a_2=0,\ b_1=1 \end{matrix}\right\} x_1.1.e^{i0} + x_2.1.e^{i\frac{\pi}{2}}$	$\left\{\begin{matrix} a_1=1,\ b_1=0 \\ a_2=1,\ b_2=1 \end{matrix}\right\} x_1.1.e^{i0} + x_2\sqrt{2}.e^{i\frac{\pi}{4}}$
9. $\left\{\begin{matrix} a_1=0,\ b_1=1 \\ a_2=0,\ b_2=0 \end{matrix}\right\} x_1.1.e^{i\frac{\pi}{2}}$	$\left\{\begin{matrix} a_1=0,\ b_1=1 \\ a_2=1,\ b_2=0 \end{matrix}\right\} x_1.1.e^{i\frac{\pi}{2}} + x_2.1.e^{i0}$
$\left\{\begin{matrix} a_1=0,\ b_1=1 \\ a_2=0,\ b_2=1 \end{matrix}\right\} x_1.1.e^{i\frac{\pi}{2}} + x_2.1.e^{i\frac{\pi}{2}}$	$\left\{\begin{matrix} a_1=0,\ b_1=1 \\ a_2=1,\ b_2=1 \end{matrix}\right\} x_1.1.e^{i\frac{\pi}{2}} + x_2\sqrt{2}.e^{i\frac{\pi}{4}}$
$\left\{\begin{matrix} a_1=1,\ b_1=1 \\ a_2=0,\ b_2=0 \end{matrix}\right\} x_1\sqrt{2}.e^{i\frac{\pi}{4}}$	$\left\{\begin{matrix} a_1=1,\ b_1=1 \\ a_2=1,\ b_2=0 \end{matrix}\right\} x_1\sqrt{2}.e^{i\frac{\pi}{4}} + x_2.1.e^{i0}$
$\left\{\begin{matrix} a_1=1,\ b_1=1 \\ a_2=0,\ b_2=0 \end{matrix}\right\} x_1\sqrt{2}.e^{i\frac{\pi}{4}} + x_2.1.e^{i\frac{\pi}{2}}$	$\left\{\begin{matrix} a_1=1,\ b_1=1 \\ a_2=1,\ b_2=1 \end{matrix}\right\} x_1\sqrt{2}.e^{i\frac{\pi}{4}} + x_2\sqrt{2}.e^{i\frac{\pi}{4}}$

or,

$$10.\quad R = \left\{\begin{array}{cccc} 0, & x_2, & ix_2, & x_2(1+i), \\ x_1, & x_1 + x_2, & x_1 + ix_2, & x_1 + x_2(1+i), \\ ix_1, & ix_1 + x_2, & ix_1 + ix_2, & ix_1 + x_2(1+i), \\ x_1(1+i), & x_1(1+i) + x_2, & x_1(1+i) + ix_2, & x_1(1+i) + x_2(1+i) \cdot \end{array}\right\}$$

The entire 16 values of R are all different, the one from the other, and also, in the general case, the entire 2^{2p} values are all different, one from the other. From these values of R we have, since $(-)^i = e^{-\pi}$,

$$11.\quad (-)^R = \left\{\begin{array}{cccc} 1, & (-)^{x_2}, & e^{-x_2\pi}, & (-)^{x_2}.e^{-x_2\pi}, \\ (-)^{x_1}, & (-)^{x_1+x_2}, & (-)^{x_1}.e^{-x_2\pi}, & (-)^{(x_1+x_2)}.e^{-x_2\pi}, \\ e^{-x_1\pi}, & (-)^{x_2}.e^{-x_1\pi}, & e^{-(x_1+x_2)\pi}, & (-)^{x_2}.e^{-(x_1+x_2)\pi}, \\ (-)^{x_1}.e^{-x_1\pi}, & (-)^{(x_1+x_2)}.e^{-x_1\pi}, & (-)^{x_1}.e^{-(x_1+x_2)\pi}, & (-)^{(x_1+x_2)}.e^{-(x_1+x_2)\pi}. \end{array}\right\}$$

From this we see that the factor $(-)^{\Sigma x_r A_r e^{i\phi_r}}$ is always either $+1$, -1, or $\pm e^{-k\pi}$, where k is an integer. In the case of the double Ξ-functions there are thus 16 functions which only change sign by replacing in the characteristic λ by $\lambda + 2x$, and so in the p-tuple Ξ-functions there will be 2^{2p} functions which will at the most only change sign by replacing λ by $\lambda + 2x$. These functions of course correspond to those values of the characteristics in which all of the b's are made zero and the a's take the values zero and unity. When this happens the Ξ-functions become the ordinary theta-functions. As the equation

$$12.\quad \Xi\binom{l_r + 2y_r}{\lambda_r + 2x_r}(v_r) = (-)^{\Sigma x_r A_r e^{i\phi_r}}.\Xi\binom{l_r}{\lambda_r}(v_r)$$

can, by virtue of the factor $(-)^{\Sigma x_r A_r e^{i\phi_r}}$ only take 2^{3p} different forms, and as in 2^{2p} of these the factor becomes either $+1$ or -1 there remain

$$2^{3p} - 2^{2p} = 2^{2p}(2^p - 1)$$

cases in which the factor becomes $= \pm e^{-k\pi}$, a constant whose greatest value for k positive is $e^{-\pi}$, or $(-)^i$.

For the theta-functions the quantities l and λ reduce to their real parts a and α, and further, a and α are integers. Observing this fact and also the fact that the summations with respect to $n_1, n_2, n_3 \ldots$ extend from $-\infty$ to $+\infty$, we readily find the number of even and odd theta-functions, that is, the number of functions which are not altered by changing the arguments $v_1, v_2 \ldots v_p$ into $-v_1, -v_2 \ldots -v_p$. The determination of the number of odd and even theta-functions is made to depend upon the fact that in the exponent of the general term of the series we can change $n + l$ into $-n - l$, where of course l is a real integer. In the present case such a change obviously cannot be made, since l is a complex quantity, viz. $l = a + ib$. It seems as if a formula of transformation

ought to be found which would represent the result of changing v into $-v$, and which would reduce to the corresponding formula for theta-functions by making b and β each zero, but I have not been able up to the present to hit upon it. I have found two methods of transforming the Ξ-functions, so that when the elements of the characteristic reduce to real quantities (integers) the corresponding theta-function formulæ are obtained. I shall only give one of them in what follows, viz. the one which, on making the b's and β's all zero, will give all of the corresponding formulæ for the theta-functions. The Ξ-function is defined by the equation

13. $$\Xi\binom{l_r}{\lambda_r}(v_r) = \sum_{n=-\infty}^{n=+\infty} \text{exp.}\binom{n_r + l_r}{v_r + \lambda_r}, \quad r = 1, 2, 3 \ldots$$

The n's are of course all even, and a single summation sign is used simply for brevity. As above we have $l = a + ib$ and $\lambda = \alpha + i\beta$, with the above restriction (which is at first not necessary) that a, b, α and β take only the values zero and unity. Now remembering that

$$\text{exp.}\binom{n_r + l_r}{v_r + \lambda_r} = \text{exp.}\left\{\frac{1}{4}(*\between n_1 + l_1,\ n_2 + l_2 \ldots n_p + l_p)^2 + \frac{1}{2}\pi i \sum_{r=1}^{r=p} (n_r + l_r)(v_r + \lambda_r)\right\},$$

and remembering that the summation with respect to n is (for even integers) from $-\infty$ to $+\infty$, we can change n into $-n$ without altering the value of the function. We now seek the value of the function

$$\Xi\binom{(-1 + ib_r)l_r}{(1 + i\beta_r)\lambda_r}(-v_r), \quad r = 1, 2, \ldots p.$$

Changing n into $-n$ the exponent of the general term becomes

$$\begin{aligned}\tfrac{1}{4}(*\between -n_1 + (-1 + ib_1)l_1, \ldots -n_p + (-1 + ib_p)l_p)^2 \\ + \tfrac{1}{2}\pi i\{[-n_1 + (-1 + ib_1)l_1][-v_1 + (1 + i\beta_1)\lambda_1] + \ldots \\ + [-n_p + (-1 + ib_p)l_p][-v_p + (1 + i\beta_p)\lambda_p]\},\end{aligned}$$

or

$$\begin{aligned}\tfrac{1}{4}(*\between n_1 + (1 - ib_1)l_1, \ldots n_p + (1 - ib_p)l_p)^2 \\ + \tfrac{1}{2}\pi i\{[n_1 + (1 - ib_1)l_1][v_1 - (1 + i\beta_1)\lambda_1] + \ldots \\ + [n_p + (1 - ib_p)\lambda_p][v_p - (1 + i\beta_p)\lambda_p]\}.\end{aligned}$$

Expanding this and subtracting from the original exponent

$$\frac{1}{4}(*\between n_1 + l_1, \ldots n_p + l_p)^2 + \frac{1}{2}\pi i \sum_{r=1}^{r=p} (n_r + l_r)(v_r + \lambda_r),$$

we have, after some simple reductions,

14. $$\tfrac{1}{4}\{-\Sigma l_r^2 - 2i\Sigma n_r b_r + \Sigma l_r^2(1-ib_r)^2 + i\Sigma\Sigma l_r b_r(n_s+l_s) - \Sigma\Sigma l_r l_s b_r b_s\}$$
$$+\tfrac{1}{2}\pi i\{\Sigma(2+i\beta_r)n_r\lambda_r + i\Sigma b_r l_r v_r + \Sigma(2+i\beta_r - ib_r + b_r\beta_r)l_r\lambda_r\},$$

the summations extending from $r=1$ to $r=p$ and in the double summations the values $r=s$ being excluded. Write this as

15. $$\tfrac{1}{4}\Phi_r + \tfrac{1}{2}\pi i\Psi_{ri};$$

if now we make $b=\beta=0$ we have

$$\Phi_r = 0 \text{ and } \Psi_r = 2\Sigma n_r\alpha_r + 2\Sigma a_r\alpha_r,$$

which is the corresponding quantity for the ϑ-functions. We have now for the Ξ-functions the relation

16. $$\Xi\begin{pmatrix}(-1+ib_r)l_r\\(1+i\beta_r)\lambda_r\end{pmatrix}(-v_r) = \Xi\begin{pmatrix}l_r\\\lambda_r\end{pmatrix}(v_r).\exp.\{\tfrac{1}{4}\Phi_r + \tfrac{1}{2}\pi i\Psi_r\}.$$

Making $b=\beta=0$ in this and we have, replacing Ξ by Θ, the well-known formula

17. $$\Theta\begin{pmatrix}a_1 a_2 \ldots a_p\\\alpha_1\alpha_2\ldots\alpha_p\end{pmatrix}(-v_1, -v_2 \ldots -v_p) = (-)^{\Sigma a_r\alpha_r}\Theta\begin{pmatrix}a_1 a_2 \ldots a_p\\\alpha_1\alpha_2\ldots\alpha_p\end{pmatrix}(v_1, v_2 \ldots v_p).$$

Another and rather more general form is obtained by changing l into $(-1+ib)l$, λ into $(1+i\beta)\lambda$ and v into $(-1+i\beta)v$. The quantity called Φ_r above undergoes no change, but we have, calling Ψ'_r the new value of Ψ_r

18. $$\Psi'_r = i\Sigma\beta_r n_r v_r + \Sigma(2+i\beta_r)n_r\lambda_r + \Sigma(ib_r + i\beta_r + b_r\beta_r)l_r v_r$$
$$+\Sigma(2+i\beta_r - ib_r + b_r\beta_r)l_r\lambda_r,$$

and

19. $$\Psi'_r - \Psi_r = i\Sigma\beta_r n_r v_r + \Sigma(i\beta_r + b_r\beta_r)l_r v_r + 2i\Sigma b_r l_r\lambda_r,$$

every term of which vanishes when l and λ are real. The quantity $\tfrac{1}{4}\Phi_r$ may be written

20. $$\tfrac{1}{4}\Phi_r = -\tfrac{i}{2}\Sigma n_r b_r - \tfrac{i}{2}\Sigma b_r l_r^2 - \tfrac{1}{4}\Sigma l_r^2 b_r^2 + \tfrac{i}{4}\Sigma\Sigma l_r b_r(n_s+l_s) - \tfrac{1}{4}\Sigma\Sigma l_r l_s b_r b_s;$$

since n is an even integer write $n=2k$, then this is

21. $$= -i\Sigma k_r b_r - \tfrac{1}{2}i\Sigma b_r l_r^2 - \tfrac{1}{4}\Sigma l_r^2 b_r^2 + \tfrac{i}{4}\Sigma\Sigma l_r b_r(2k_s+l_s) - \tfrac{1}{4}\Sigma\Sigma l_r l_s b_r b_s;$$

again since $b=0$ or 1 the first term of this becomes a real multiple of i, and the remaining terms become quantities of the form $M+iN$, where M and N are real and of the forms μ, $\frac{\nu}{2}$, $\frac{\rho}{4}$, where μ, ν and ρ are real integers. Similar remarks might be made concerning Ψ_r, but, for the present, it is not necessary to consider this quantity We may make one further transformation similar to the above and to the corresponding transformation in the case of the theta-functions. For simplicity take the function

22. $$\Xi\begin{pmatrix}a_1+ib_1, & a_2+ib_2\\\alpha_1+i\beta_1, & \alpha_2+i\beta_2\end{pmatrix}(v_1, v_2) = \Sigma\Sigma\exp.[\tfrac{1}{4}(a_{11}, a_{12}, a_{22})(n_1+a_1+ib_1, n_2+a_2+ib_2)^2$$
$$+\tfrac{1}{2}\pi i\{(n_1+a_1+ib_1)(v_1+\alpha_1+i\beta_1) + (n_2+a_2+ib_2)(v_2+\alpha_2+i\beta_2)\}].$$

Now we can change n_1 and n_2 into $-n_1$ and $-n_2$ without altering the value of the function; make this change and also change l_1 and l_2 into $-l_1$ and $-l_2$ and v_1 and v_2 into $-v_1$ and $-v_2$. We have then to find the value of the function,

$$\Xi\begin{pmatrix}-l_1 & -l_2\\ \lambda_1 & \lambda_2\end{pmatrix}(-v_1, -v_2).$$

The exponent is, on changing n into $-n$,

$$\tfrac{1}{4}(*\between -n_1-l_1, -n_2-l_2)^2+\tfrac{1}{2}\pi i\{(-n_1-l_1)(-v_1+\lambda_1)+(-n_2-l_2)(-v_2+\lambda_2)\}.$$

The quadratic term is unaltered, but the linear term becomes

$$\tfrac{1}{2}\pi i\{(n_1+l_1)(v_1+\lambda_1)+(n_2+l_2)(v_2+\lambda_2)\}-\pi i\{(n_1+l_1)\lambda_1+(n_2+l_2)\lambda_2\};$$

this differs from the original exponent by the term,

23. $$-\pi i(n_1\lambda_1+n_2\lambda_2)-\pi i(l_1\lambda_1+l_2\lambda_2),$$

say,

23′. $$-\pi i N-\pi i L.$$

Make now $n=2k$ and give λ_1 and λ_2 all their values: these are

$$\lambda_1=0,\ 1,\ i,\ 1+i,$$
$$\lambda_2=0,\ 1,\ i,\ 1+i;$$

then all possible values of N are

24.
$$\begin{array}{cccc}
0, & 2k_2, & 2ik_2, & 2k_2+2ik_2,\\
2k_1, & 2k_1+2k_2, & 2k_1+2ik_2, & 2k_1+2k_2+2ik_2,\\
2ik_1, & 2ik_1+2k_2, & 2ik_1+2ik_2, & 2ik_1+2ik_2+2k_2,\\
2k_1+2ik_1, & 2k_1+2k_2+2ik_2, & 2k_1+2ik_1+2ik_2, & 2k_1+2k_2+2ik_1+2ik_2.
\end{array}$$

(Notice in forming this table that after writing down the first line and first column all the remaining terms are formed by combining them: viz., the second, third and fourth elements in the second column are obtained by adding the first element in this column to the second, third and fourth elements in the first column respectively, etc.). The exponential thus takes one of the values,

25. $$+1,\ e^{2\tau\pi},$$

where τ is a positive or negative integer, when λ reduces to its real part 0 or 1 this exponential is $=+1$. There are in all 256 different values of L, 16 of which for l and $\lambda=0, 1$, make the exponential $e^{-\pi i(l_1\lambda_1+l_2\lambda_2)}$ equal to ± 1, ten values giving $+1$ and six values giving -1. We have finally

26. $$\Xi\begin{pmatrix}-l_1 & -l_2\\ \lambda_1 & \lambda_2\end{pmatrix}(-v_1, -v_2)$$
$$=(-)^{l_1\lambda_1+l_2\lambda_2}\Sigma\Sigma(1)^{k_1l_1+k_2l_2}\exp.\{\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between n_1+l_1, n_2+l_2)^2$$
$$+\tfrac{1}{2}\pi i[(n_1+l_1)(v_1+\lambda_1)+(n_2+l_2)(v_2+\lambda_2)]\}.$$

For the general case we have obviously

$$27. \qquad \Xi\binom{-l_1 \dots -l_p}{\lambda_1 \dots \ \lambda_p}(-v_1, \dots, -v_p) = (-)^{\Sigma l_r \lambda_r}\Sigma \dots \Sigma(1)^{\Sigma k_r l_r}$$

$$\times \text{ exp.}\left\{\tfrac{1}{4}(*)(n_1+l_1, \dots n_p+l_p)^2 + \tfrac{1}{2}\pi i \Sigma(n_r+l_r)(v_r+\lambda_r)\right\}.$$

For $p=2$ write $l_1\lambda_1 + l_2\lambda_2 = \sigma$; the entire set of values of σ are given in the following table:

28.

0	0	0	0												
0	1	i	$1+i$		A				A				A		
0	i	-1	$-1+i$												
0	$1+i$	$-1+i$	$2i$												
				1	1	1	1	i	i	i	i	$1+i$	$1+i$	$1+i$	$1+i$
	A			1	2	$1+i$	$2+i$	i	$1+i$	$2i$	$1+2i$	$1+i$	$2+i$	$1+2i$	$2+2i$
				1	$1+i$	0	i	i	$2i$	$-1+i$	$-1+2i$	$1+i$	$1+2i$	i	$2i$
				1	$2+i$	i	$1+2i$	i	$1+2i$	$-1+2i$	$3i$	$1+i$	$2+2i$	$2i$	$1+3i$
				i	i	i	i	-1	-1	-1	-1	$-1+i$	$-1+i$	$-1+i$	$-1+i$
	A			i	$1+i$	$2i$	$1+2i$	-1	0	$-1+i$	i	$-1+i$	i	$-1+2i$	$2i$
				i	$2i$	$-1+i$	$-1+2i$	-1	$-1+i$	-2	$-2+i$	$-1+i$	$-1+2i$	$-2+i$	$-2+2i$
				i	$1+2i$	$-1+2i$	$3i$	-1	i	$-2+i$	$-1+2i$	$-1+i$	$2i$	$-2+2i$	$-1+3i$
				$1+i$	$1+i$	$1+i$	$1+i$	$-1+i$	$-1+i$	$-1+i$	$-1+i$	$2i$	$2i$	$2i$	$2i$
	A			$1+i$	$2+i$	$1+2i$	$2+2i$	$-1+i$	i	$-1+2i$	$2i$	$2i$	$1+2i$	$3i$	$1+3i$
				$1+i$	$1+2i$	i	$2i$	$-1+i$	$-1+2i$	$-2+i$	$-2+2i$	$2i$	$3i$	$-1+2i$	$-1+3i$
				$1+i$	$2+2i$	$2i$	$1+3i$	$-1+i$	$2i$	$-2+2i$	$-1+3i$	$2i$	$1+3i$	$-1+3i$	$4i$

The spaces marked A are to be filled up in the same way as the upper left-hand corner space (each of the sixteen spaces is seen to be symmetrically arranged, and also the entire group is seen to be symmetrical). There are in all only 19 *different* values of σ, viz.

$0,\ 1,\ 2,\ -1,\ -2,\ i,\ 1+i,\ -1+i,\ 2i,\ 2+i,\ -2+i,\ 1+2i,$
$-1+2i,\ 3i,\ 2+2i,\ 1+3i,\ -2+2i,\ -1+3i,\ 4i.$

The number of times that each of these appears is:

0	number of times	51	$-2+i$	number of times	4
1	" "	14	$1+2i$	" "	10
2	" "	1	$-1+2i$	" "	10
-1	" "	14	$3i$	" "	4
-2	" "	1	$2+2i$	" "	4
i	" "	36	$1+3i$	" "	4
$1+i$	" "	32	$-2+2i$	" "	4
$-1+i$	" "	32	$-1+3i$	" "	4
$2i$	" "	26	$4i$	" "	1
$2+i$	" "	4			

making the proper total of 256. The above formula of transformation is more convenient when written in the form,

29. $$\Xi\begin{pmatrix}-l_1 \dots -l_p \\ \lambda_1 \dots \lambda_p\end{pmatrix}(-v_1, \dots -v_p) = (-)^{\Sigma l_r \lambda_r}\Sigma \dots \Sigma(-)^{\Sigma n_r l_r}$$
$$\times \exp.\left\{\tfrac{1}{4}(*)(n_1+l_1, \dots n_p+l_p)^2 + \tfrac{1}{2}\pi i\Sigma(n_r+l_r)(v_r+\lambda_r)\right\}.$$

The values (for $p=2$) of $n_1 l_1 + n_2 l_2$ are of course obtained by replacing k_1 and k_2 in (24) by n_1 and n_2. We have then for $n_1 l_1 + n_2 l_2$ the table:

30.

$l_1=0,\ l_2=0$ 0	$l_1=1,\ l_2=0$ n_1	$l_1=i,\ l_2=0$ in_1	$l_1=1+i,\ l_2=0$ n_1+in_1
$l_1=0,\ l_2=1$ n_2	$l_1=1,\ l_2=1$ n_1+n_2	$l_1=i,\ l_2=1$ n_1+in_2	$l_1=1+i,\ l_2=1$ $n_1+n_2+in_2$
$l_1=0,\ l_2=i$ in_2	$l_1=1,\ l_2=i$ n_1+in_2	$l_1=i,\ l_2=i$ in_1+in_2	$l_1=1+i,\ l_2=i$ $in_1+in_2+in_2$
$l_1=0,\ l_2=1+i$ n_2+in_2	$l_1=1,\ l_2=1+i$ $n_1+n_2+in_2$	$l_1=i,\ l_2=1+i$ $in_1+in_2+n_2$	$l_1=1+i,\ l_2=1+i$ $n_1+n_2+in_1+in_2$

The upper line in each block gives the values of l_1 and l_2, and the lower line the corresponding value of $n_1 l_1 + n_2 l_2$.

For greater convenience, table (28) may be presented in the following form. Each square contains all of the values of $l_1\lambda_1 + l_2\lambda_2$ corresponding to the values of l_1 and l_2 written above the square.

31.

$l_1=0$, $l_2=0$				$l_1=1$, $l_2=0$				$l_1=i$, $l_2=0$				$l_1=1+i$, $l_2=0$			
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	1	1	i	i	i	i	$1+i$	$1+i$	$1+i$	$1+i$
0	0	0	0	i	i	i	i	-1	-1	-1	-1	$-1+i$	$-1+i$	$-1+i$	$-1+i$
0	0	0	0	$1+i$	$1+i$	$1+i$	$1+i$	$-1+i$	$-1+i$	$-1+i$	$-1+i$	$2i$	$2i$	$2i$	$2i$
$l_1=0$, $l_2=1$				$l_1=1$, $l_2=1$				$l_1=i$, $l_2=1$				$l_1=1+i$, $l_2=1$			
0	1	i	$1+i$	0	1	i	$1+i$	0	1	i	$1+i$	0	1	i	$1+i$
0	1	i	$1+i$	1	2	$1+i$	$2+i$	i	$1+i$	$2i$	$1+2i$	$1+i$	$2+i$	$1+2i$	$2+2i$
0	1	i	$1+i$	i	$1+i$	$2i$	$1+2i$	-1	0	$-1+i$	i	$-1+i$	i	$-1+2i$	$2i$
0	1	i	$1+i$	$1+i$	$2+i$	$1+2i$	$2+2i$	$-1+i$	i	$-1+2i$	$2i$	$2i$	$1+2i$	$3i$	$1+3i$
$l_1=0$, $l_2=i$				$l_1=1$, $l_2=i$				$l_1=i$, $l_2=i$				$l_1=1+i$, $l_2=i$			
0	i	-1	$-1+i$	0	i	-1	$-1+i$	0	i	-1	$-1+i$	0	i	-1	$-1+i$
0	i	-1	$-1+i$	1	$1+i$	0	i	i	$2i$	$-1+i$	$-1+2i$	$1+i$	$1+2i$	i	$2i$
0	i	-1	$-1+i$	i	$2i$	$-1+i$	$-1+2i$	-1	$-1+i$	-2	$-2+i$	$-1+i$	$-1+2i$	$-2+i$	$-2+2i$
0	i	-1	$-1+i$	$1+i$	$1+2i$	i	$2i$	$-1+i$	$-1+2i$	$-2+i$	$-2+2i$	$2i$	$3i$	$-1+2i$	$-1+3i$
$l_1=0$, $l_2=1+i$				$l_1=1$, $l_2=1+i$				$l_1=i$, $l_2=1+i$				$l_1=1+i$, $l_2=1+i$			
0	$1+i$	$-1+i$	$2i$	0	$1+i$	$-1+i$	$2i$	0	$1+i$	$-1+i$	$2i$	0	$1+i$	$-1+i$	$2i$
0	$1+i$	$-1+i$	$2i$	1	$2+i$	i	$1+2i$	i	$1+2i$	$-1+2i$	$3i$	$1+i$	$2+2i$	$2i$	$1+3i$
0	$1+i$	$-1+i$	$2i$	i	$1+2i$	$-1+2i$	$3i$	-1	i	$-2+i$	$-1+2i$	$-1+i$	$2i$	$-2+2i$	$-1+3i$
0	$1+i$	$-1+i$	$2i$	$1+i$	$2+2i$	$2i$	$1+3i$	$-1+i$	$2i$	$-2+2i$	$-1+3i$	$2i$	$1+3i$	$-1+3i$	$4i$

A simple case which might be considered is when l is a real integer, as in this case the factors $(-)^{\Sigma nl}$ all reduce to unity, and the l's in the characteristic may obviously be written with the plus sign.

PERIODS.

We may still, for simplicity, continue to deal with the double Ξ-functions, as the general properties which are being studied are capable of immediate generalization to the case of p-tuple functions. From the definition of these functions we see at once that if x_1 and x_2 denote any two integers we must have

32. $$\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1+x_1,\ v_2+x_2) = \Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1+x_1 & \lambda_2+x_2 \end{pmatrix}(v_1,\ v_2),$$

that is, the effect of altering the arguments v_1 and v_2 by the addition of the integers x_1 and x_2 only interchanges the functions. If, however, we write $2x_1$ and $2x_2$ instead of x_1 and x_2 we have

33. $$\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1 + 2x_1,\ v_2 + 2x_2) = \Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 + 2x_1 & \lambda_2 + 2x_2 \end{pmatrix}(v_1,\ v_2)$$

or from equation 7,

34. $$\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1 + 2x_1,\ v_2 + 2x_2) = (-)^{l_1 x_1 + l_2 x_2}\,\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1,\ v_2)$$

and the factor $(-)^{l_1 x_1 + l_2 x_2}$ has been shown to be either

$$+1,\ -1,\ +e^{-k\tau},\ -e^{-k\tau},$$

where k is an integer; so the function is either unaltered, altered in sign, or multiplied by one of the constants $\pm e^{-k\tau}$. The cases where there is no alteration or at most a change of sign, correspond to l_1 and l_2 real integers. If we replace x_1 and x_2 by $4x_1$ and $4x_2$ respectively, x_1 and x_2 still integers, we have

35. $$\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1 + 4x_1,\ v_2 + 4x_2) = (-)^{2(l_1 x_1 + l_2 x_2)}\,\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1,\ v_2).$$

The factor $(-)^{2(l_1 x_1 + l_2 x_2)}$ is in this case either $+1$ or $e^{-2k\pi}$, where k is an integer. There is thus in the case of the Ξ-functions no period in the ordinary sense of the word, those functions only possess a period for which l_1 and l_2 are real. There are $64, = 2^6$ functions which have periods, and $192, = 2^8 - 2^6$ which are multiplied by the above factor; in general there are 2^{3p} functions which possess a true period and $2^{3p}(2^p - 1)$ functions which are multiplied by an exponential factor. The *quasi-periods* are obtained just as in the case of the theta-functions, and to show their existence I shall merely reproduce, with the necessary changes of notation, Prof. Cayley's proof in § 10 of his memoir. Taking x_1 and x_2 integers, we consider the effect of the change of v_1, v_2 into

$$v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2),\ v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2).$$

Starting from the function

$$\Xi\begin{pmatrix} l_1 - x_1 & l_2 - x_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}\left(v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2),\ v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right)$$

we have for the argument of the exponential

$$\begin{aligned} &\tfrac{1}{4}(a_{11},\ a_{12},\ a_{22})(n_1 + l_1 - x_1,\ n_2 + l_2 - x_2)^2 \\ &\quad + \tfrac{1}{2}\pi i\left\{(n_1 + l_1 - x_1)\left(v_1 + \lambda_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2)\right)\right. \\ &\qquad\qquad \left. + (n_2 + l_2 - x_2)\left(v_2 + \lambda_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right)\right\}, \end{aligned}$$

this is

$$\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between n_1 + l_1, n_2 + l_2)^2 + \tfrac{1}{2}\pi i\big((n_1 + l_1)(v_1 + \lambda_1) + (n_2 + l_2)(v_2 + \lambda_2)\big)$$

(the original exponent) + other terms which are as follows: viz. they are

$$-\tfrac{1}{2}(a_{11}, a_{12}, a_{22}\between n_1 + l_1, n_2 + l_2\between x_1, x_2) + \tfrac{1}{2}[(n_1 + l_1)(a_{11}x_1 + a_{12}x_2) + (n_2 + l_2)(a_{12}x_1 + a_{22}x_2)]$$

$$+\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between x_1, x_2)^2 \qquad -\tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)]$$

$$-\tfrac{1}{2}[x_1(a_{11}x_1 + a_{12}x_2) + x_2(a_{12}x_1 + a_{22}x_2)].$$

The terms in the right-hand column reduce at once to

$$+\tfrac{1}{2}(a_{11}, a_{12}, a_{22}\between n_1 + l_1, n_2 + l_2\between x_1, x_2)$$
$$-\tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)]$$
$$-\tfrac{1}{2}[a_{11}, a_{12}, a_{22}\between x_1, x_2)^2.$$

The original exponent is then increased by the terms

$$-\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between x_1, x_2)^2 - \tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)],$$

which are independent of n_1 and n_2, and they thus affect each term of the series with the same exponential factor. The result is

$$\Xi\begin{pmatrix} l_1 - x_1, & l_2 - x_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}\left(v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2),\ v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right)$$

36.

$$= \exp.\{-\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between x_1, x_2)^2 - \tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)]\}\,\Xi\begin{pmatrix} l_1 & l_2 \\ \lambda_2 & \lambda_2 \end{pmatrix}(v_1, v_2);$$

or what is the same thing, replacing l_1 and l_2 by $l_1 + x_1$ and $l_2 + x_2$ (this change, since x_1 and x_2 are integers, only alters the real part of l_1 and l_2 and so is admissible),

$$\Xi\begin{pmatrix} l_1, & l_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}\left(v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2),\ v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2)\right)$$

37.

$$= \exp.\{-\tfrac{1}{4}(a_{11}, a_{12}, a_{22}\between x_1, x_2)^2 - \tfrac{1}{2}\pi i[x_1(v_1 + \lambda_1) + x_2(v_2 + \lambda_2)]\}$$
$$\Xi\begin{pmatrix} l_1 + x_1, & l_2 + x_2 \\ \lambda_1 & \lambda_2 \end{pmatrix}(v_1, v_2).$$

If x_1 and x_2 are even, say $2x_1$ and $2x_2$, the function on the right-hand side of this equation becomes,

38.

$$\Xi\begin{pmatrix} l_1 + 2x_1, & l_2 + 2x_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}(v_1, v_2), = \Xi\begin{pmatrix} l_2, & l_2 \\ \lambda_1, & \lambda_2 \end{pmatrix}(v_1, v_2),$$

but the exponential factor still remains. The formulæ show that the effect of changing v_1 and v_2 into

$$v_1 + \frac{1}{\pi i}(a_{11}x_1 + a_{12}x_2),\ v_2 + \frac{1}{\pi i}(a_{12}x_1 + a_{22}x_2),$$

where x_1 and x_2 are integers, is to interchange the functions and to affect each of them with an exponential factor; and we may say, as in the case of the ϑ-functions, that $\frac{1}{\pi i}(a_{11}, a_{12})$, $\frac{1}{\pi i}(a_{12}, a_{22})$ are *conjoint quarter quasi-periods.* We

may also say that the Ξ-functions have the quarter-periods $(1, 1)$, the half-periods $(2, 2)$ and the whole periods $(4, 4)$, understanding that only those functions are truly periodic for which the l's are all real, *i. e.* for one case all zero or unity.

THE PRODUCT-THEOREM.

If we multiply together two *theta-functions*, say

$$\vartheta\binom{l_1,\ l_2}{\lambda_1,\ \lambda_2}(v_1+v'_1,\ v_2+v'_2),\ \vartheta\binom{l'_1,\ l'_2}{\lambda'_1,\ \lambda'_2}(v_1-v'_1,\ v_2-v'_2)$$

we have, for the result, the sum of four products of the form,

$$\Theta\binom{\frac{1}{2}(l_1+l'_1)+p_1,\ \frac{1}{2}(l_2+l'_2)+p_2}{\lambda_1+\lambda'_1,\qquad \lambda_2+\lambda'_2}(2v_1,\ 2v_2).\Theta\binom{\frac{1}{2}(l_1-l'_1)+p_1,\ \frac{1}{2}(l_2-l'_2)+p_2}{\lambda_1-\lambda'_1,\qquad \lambda_2-\lambda'_2}(2v'_1,\ 2v'_2),$$

where p_1 and p_2 take the values 0 and 1, and where Θ is written to denote that a_{11}, a_{12}, a_{22} are replaced by $2a_{11}$, $2a_{12}$, $2a_{22}$. The proof given of this theorem is wholly independent of the nature of l_1 and l_2 and so holds for the case of the double Ξ-functions. Using here ξ to denote that the parameters a_{11}, a_{12}, a_{22} have been changed into $2a_{11}$, $2a_{12}$, $2a_{22}$ we have

$$\begin{aligned}
39.\qquad & \Xi\binom{l_1,\ l_2}{\lambda_1,\ \lambda_2}(v_1+v'_1).\Xi\binom{l'_1,\ l'_2}{\lambda'_1,\ \lambda'_2}(v_1-v'_1)\\
&= \xi\binom{\frac{1}{2}(l_1+l'_1),\ \frac{1}{2}(l_2+l'_2)}{\lambda_1+\lambda'_1,\quad \lambda_2+\lambda'_2}(2v_1).\xi\binom{\frac{1}{2}(l_1-l'_1),\ \frac{1}{2}(l_2-l'_2)}{\lambda_1-\lambda'_1,\quad \lambda_2-\lambda'_2}(2v'_1)\\
&+ \xi\binom{\frac{1}{2}(l_1+l'_1)+1,\ \frac{1}{2}(l_1+l'_2)}{\lambda_1+\lambda'_1\qquad \lambda_2+\lambda'_2}(2v_1).\xi\binom{\frac{1}{2}(l_1-l'_1)+1,\ \frac{1}{2}(l_2-l'_2)}{\lambda_1-\lambda'_1\qquad \lambda_2-\lambda'_2}(2v'_1)\\
&+ \xi\binom{\frac{1}{2}(l_1+l'_1),\ \frac{1}{2}(l_2+l'_2)+1}{\lambda_1+\lambda'_1\quad \lambda_2+\lambda'_2}(2v_1).\xi\binom{\frac{1}{2}(l_1-l'_1),\ \frac{1}{2}(l_2-l'_2)+1}{\lambda_1-\lambda'_1\quad \lambda_2-\lambda'_2}(2v'_1)\\
&+ \xi\binom{\frac{1}{2}(l_1+l'_1)+1,\ \frac{1}{2}(l_2+l'_2)+1}{\lambda_1+\lambda'_1,\qquad \lambda_2+\lambda'_2}(2v_1).\xi\binom{\frac{1}{2}(l_1-l'_1)+1,\ \frac{1}{2}(l_2-l'_2)+1}{\lambda_1-\lambda'_1,\qquad \lambda_2-\lambda'_2}(2v'_1)
\end{aligned}$$

the arguments $v_1+v'_1$, $v_1-v'_1$ are written for brevity instead of $(v_1+v'_1,\ v_2+v'_2)$ and $(v_1-v'_1,\ v_2-v'_2)$, and similarly on the right-hand side of the equation one argument only is exhibited.

In order to develop the results it would be necessary, or at least desirable, to form two tables, one giving all the values of the upper line of the new characteristics, and the other giving all the values of the lower line of the same. This is what Prof. Cayley has done in his memoir on the double theta-functions. The tables are so long, however, that I do not give them here, particularly as their application to form a complete product table for the double Ξ-functions would give an exceedingly long table. There are in all 2^{16} (in general 2^{8p}) different products in the case of the double Ξ-functions; these consist of a *square-set* of 256 $(=2^{4p})$ products and 255 other sets, each containing 256 products. Using

suffixes for a moment to denote the different Ξ-functions, we define a *square-set* as a set consisting of products of the form,

$$\Xi_{ik}(v_1+v'_1,\, v_2+v'_2)\,.\,\Xi_{ik}(v_1-v'_1,\, v_2-v'_2)\,;$$

the remaining sets would have different suffixes for each Ξ.

All of the theorems which have been here set down for the double Ξ-functions are, as has been already remarked, capable of immediate generalization to the case of the p-tuple functions. An alteration of the l's by an even integer produces no change in the function; an alteration of the λ's by an even integer gives a formula precisely like that in the case of the ϑ-functions, the only difference being that the factor, say $(-)^{\delta}$, which is introduced, does not take merely the values $+1$ and -1, but the values $+1, -1, +e^{-\tau\pi}, -e^{-\tau\pi}$ where τ is an integer. An alteration of l or λ by an odd integer interchanges the functions (it would be interesting to examine the effect of increasing l or λ or both by a complex quantity; I shall take this up later). The change of v into $-v$ gives for 2^{2p} functions formulæ which are identical with those for the theta-functions. For the remaining $2^{2p}(2^{2p}-1)$ functions there is a factor under the sign of summation which depends for its value upon the value of n, and the factor outside the sign of summation is $+1, -1, -e^{-\tau\pi}, +e^{-\tau\pi}$.

By increasing the v's by $x_1, x_2 \ldots x_p$ respectively, we only interchange the functions (this of course is the same as increasing the λ's by the same integers); increasing each v by $2x$ we multiply the function by $(-)^{\Sigma lx}$, a factor which takes the values $+1, -1, -e^{-\tau\pi}, +e^{-\tau\pi}$; finally, increasing each v by $4x$ we multiply the function by $(-)^{2\Sigma lx}$, so that we have 2^{3p} functions which are unchanged by this alteration of the arguments v and $2^{3p}(2^p-1)$ which are multiplied by a factor $e^{-\tau\pi}$.

I shall proceed now to a more particular examination of the Ξ-functions for the case $p=1$. There are sixteen functions corresponding to this value of p, the characteristics for which are

$$40. \qquad \begin{matrix} \binom{0}{0} & \binom{1}{0} & \binom{0}{1} & \binom{1}{1} \\ \binom{i}{0} & \binom{1+i}{0} & \binom{i}{1} & \binom{1+i}{1} \\ \binom{0}{i} & \binom{1}{i} & \binom{0}{1+i} & \binom{1}{1+i} \\ \binom{i}{i} & \binom{1+i}{i} & \binom{i}{1+i} & \binom{1+i}{1+i}. \end{matrix}$$

The first line of 40 gives the ordinary theta-functions of one variable corresponding to the elliptic functions. For the first line we have $b=\beta=0$; for the second line $b=1$, $\beta=0$; for the third line $b=0$, $\beta=1$; for the fourth line $b=\beta=1$; in each of the four lines a and α take the values 0 and 1.

The Ξ-function of one variable is defined by the equation

$$\Xi\begin{pmatrix} l \\ \lambda \end{pmatrix}(v)=\Sigma e^{\frac{1}{4}\mathrm{a}(n+l)^2+\frac{1}{2}\pi i(n+l)(v+\lambda)}$$

where $l=a+ib$ and $\lambda=\alpha+i\beta$, a, b, α, β taking as above the values 0 and 1; a is also written instead of a_{11} in order to avoid suffixes. The 16 different functions will be denoted by the symbols

$$\begin{array}{cccc} \vartheta_1 & \vartheta_2 & \vartheta_3 & \vartheta_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{array}$$

viz., we have, omitting for convenience the argument on the left-hand side,

$$\vartheta_1=\Xi\begin{pmatrix} 0 \\ 0 \end{pmatrix}(v),\quad \vartheta_2=\Xi\begin{pmatrix} 1 \\ 0 \end{pmatrix}(v),\quad \vartheta_3=\Xi\begin{pmatrix} 0 \\ 1 \end{pmatrix}(v),\quad \vartheta_4=\Xi\begin{pmatrix} 1 \\ 1 \end{pmatrix}(v),$$

$$\xi_1=\Xi\begin{pmatrix} i \\ 0 \end{pmatrix}(v),\quad \xi_2=\Xi\begin{pmatrix} 1+i \\ 0 \end{pmatrix}(v),\quad \xi_3=\Xi\begin{pmatrix} i \\ 1 \end{pmatrix}(v),\quad \xi_4=\Xi\begin{pmatrix} 1+i \\ 1 \end{pmatrix}(v),$$

$$\eta_1=\Xi\begin{pmatrix} 0 \\ i \end{pmatrix}(v),\quad \eta_2=\Xi\begin{pmatrix} 1 \\ i \end{pmatrix}(v),\quad \eta_3=\Xi\begin{pmatrix} 0 \\ 1+i \end{pmatrix}(v),\quad \eta_4=\Xi\begin{pmatrix} 1 \\ 1+i \end{pmatrix}(v),$$

$$\zeta_1=\Xi\begin{pmatrix} i \\ i \end{pmatrix}(v),\quad \zeta_2=\Xi\begin{pmatrix} 1+i \\ i \end{pmatrix}(v),\quad \zeta_3=\Xi\begin{pmatrix} i \\ 1+i \end{pmatrix}(v),\quad \zeta_4=\Xi\begin{pmatrix} 1+i \\ 1+i \end{pmatrix}(v).$$

Writing $e^{\mathrm{a}}=q$ and converting the exponentials into circular functions we have the known formulæ

$$\vartheta_1=1+2\sum_{r=1}^{r=\infty} q^{r^2}\cos r\pi v,\qquad \vartheta_2=2\sum_{r=0}^{r=\infty} q^{\frac{(2r+1)^2}{4}}\cos\frac{2r+1}{2}\pi v,$$

$$\vartheta_3=1+2\sum_{r=1}^{r=\infty}(-)^r q^{r^2}\cos r\pi v,\quad \vartheta_4=2\sum_{r=0}^{r=\infty}(-)^{2r+1}q^{\frac{(2r+1)^2}{4}}\sin\frac{2r+1}{2}\pi v.$$

Making the same substitution in the case of the functions ξ_1, ξ_2 . . . ζ_4, and writing $n=2r$ we have

$$\xi_1=e^{-\frac{\pi v}{2}}\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)}\{\cos\pi r v(q^{ri}+q^{-ri})+i\sin\pi r v(q^{ri}-q^{-ri})\}$$

$$\xi_2 = e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{r(r+1)} \left\{ \cos \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} + q^{-i\frac{2r+1}{2}} \right) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} - q^{-i\frac{2r+1}{2}} \right) \right\}$$

$$\xi_3 = e^{-\frac{\pi(v+1)}{2}} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \{ \cos \pi r v (q^{ri} + q^{-ri}) + i \sin \pi r v (q^{ri} - q^{-ri}) \}$$

$$\xi_4 = i e^{-\frac{\pi(v+1)}{2}} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \left\{ \cos \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} + q^{-i\frac{2r+1}{2}} \right) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v \left(q^{i\frac{2r+1}{2}} + q^{i\frac{2r+1}{2}} \right) \right\}$$

$$\eta_1 = \sum_{r=0}^{r=\infty} q^{r^2} \{ \cos r\pi v (e^{r\pi} + e^{-r\pi}) - i \sin r\pi v (e^{r\pi} - e^{-r\pi}) \}$$

$$\eta_2 = \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(2r+1)^2} \left\{ \cos \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} + e^{-\frac{2r+1}{2}\pi} \right) \right.$$
$$\left. - i \sin \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} - e^{-\frac{2r+1}{2}\pi} \right) \right\}$$

$$\eta_3 = \sum_{r=0}^{r=\infty} (-)^r q^{r^2} \{ \cos r\pi v (e^{r\pi} + e^{-r\pi}) - i \sin r\pi v (e^{r\pi} - e^{-r\pi}) \}$$

$$\eta_4 = i \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(2r+1)^2} \left\{ - \cos \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} - e^{-\frac{2r+1}{2}\pi} \right) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v \left(e^{\frac{2r+1}{2}\pi} + e^{-\frac{2r+1}{2}\pi} \right) \right\}$$

$$\zeta_1 = - i e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} e^{r\pi} q^{ri} q^{\frac{1}{4}(4r^2-1)} \{ \cos r\pi v (e^{-2r\pi} + q^{-2ri}) + i \sin r\pi v (e^{-2r\pi} - q^{-2ri}) \}$$

$$\zeta_2 = - i e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} e^{\frac{(2r+1)\pi}{2}} . q^{\frac{(2r+1)i}{2}} . q^{r(r+1)} \left\{ \cos \frac{(2r+1)\pi v}{2} (e^{-(2r+1)\pi} + q^{-(2r+1)i}) \right.$$
$$\left. + i \sin \frac{(2r+1)\pi v}{2} (e^{-(2r+1)\pi} - q^{-(2r+1)i}) \right\}$$

$$\zeta_3 = - i e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} e^{r\pi} q^{ri} \{ \cos r\pi v (e^{-2r\pi} + q^{-2ri})$$
$$+ i \sin r\pi v (e^{-2r\pi} - q^{-2ri}) \}$$

$$\zeta_4 = e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} e^{\frac{\pi}{2}(2r+1)} q^{\frac{1}{2}(2r+1)i} \left\{ \cos \frac{2r+1}{2} \pi v (e^{-\pi(2r+1)} - e^{-(2r+1)i}) \right.$$
$$\left. + i \sin \frac{2r+1}{2} \pi v (e^{-\pi(2r+1)} + e^{-(2r+1)i}) \right\}$$

The formulæ for $\xi_1, \xi_2 \ldots \zeta_4$ may be written in a little different form; viz., writing for q its value e^{a} we have

$$\xi_1 = 2e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r(\pi v + a)$$

$$\xi_2 = 2e^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{r(r+1)} \cos \frac{2r+1}{2}(\pi v + a)$$

$$\xi_3 = 2e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r(\pi v + a)$$

$$\xi_4 = -2e^{-\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin \frac{2r+1}{2}(\pi v + a)$$

$$\eta_1 = 2 \sum_{r=0}^{r=\infty} q^{r^2} \cos r\pi(v - i)$$

$$\eta_2 = 2 \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(2r+1)^2} . \cos \frac{2r+1}{2} \pi(v - i)$$

$$\eta_3 = 2 \sum_{r=0}^{r=\infty} (-)^r q^{r^2} \cos r\pi(v - i)$$

$$\eta_4 = -2 \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(2r+1)^2} \sin \frac{2r+1}{2} \pi(v - i)$$

$$\zeta_1 = -2ie^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r(\pi(v+i) + a)$$

$$\zeta_2 = -2ie^{-\frac{\pi v}{2}} \sum_{r=0}^{r=\infty} q^{r(r+1)} \cos \frac{2r+1}{2}(\pi(v+i) + a)$$

$$\zeta_3 = -2ie^{\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r(\pi(v+i) + a)$$

$$\zeta_4 = 2ie^{\frac{\pi}{2}(v+1)} \sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin \frac{2r+1}{2}(\pi(v+i) + a)$$

The notation of hyperbolic functions might also have been employed for the

representation of the functions $\eta_1, \eta_2 \dots \zeta_4$. The constants derived from these functions by placing $v=0$ are as follows:

$$\vartheta_1(0) = 1 + 2\sum_{r=0}^{r=\infty} q^{r^2}, \qquad \vartheta_2(0) = 2\sum_{r=0}^{r=\infty} q^{\frac{(2r+1)^2}{4}}$$

$$\vartheta_3(0) = 1 + 2\sum_{r=0}^{r=\infty} (-)^r q^{r^2}, \qquad \vartheta_4(0) = 0$$

$$\xi_1(0) = 2\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r\mathrm{a}, \qquad \xi_2(0) = 2\sum_{r=0}^{r=\infty} q^{r(r+1)} \cos\frac{2r+1}{2}\mathrm{a}$$

$$\xi_3(0) = 2e^{-\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r\mathrm{a}, \qquad \xi_4(0) = -2e^{-\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin\frac{2r+1}{2}\mathrm{a}$$

$$\eta_1(0) = 2\sum_{r=0}^{r=\infty} q^{r^2} \cosh r\pi \qquad \eta_2(0) = 2\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(2r+1)^2} \cosh\frac{2r+1}{2}\pi$$

$$\eta_3(0) = 2\sum_{r=0}^{r=\infty} (-)^r q^{r^2} \cosh r\pi \qquad \eta_4(0) = 2i\sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(2r+1)^2} \sinh\frac{2r+1}{2}\pi$$

$$\zeta_1(0) = -2i\sum_{r=0}^{r=\infty} q^{\frac{1}{4}(4r^2-1)} \cos r(\pi i + \mathrm{a}), \quad \zeta_2(0) = -2i\sum_{r=0}^{r=\infty} q^{r(r+1)} \cos\frac{2r+1}{2}(\pi i + \mathrm{a})$$

$$\zeta_3(0) = -2ie^{\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{\frac{1}{4}(4r^2-1)} \cos r(\pi i + \mathrm{a}),$$

$$\zeta_4(0) = 2ie^{\frac{\pi}{2}}\sum_{r=0}^{r=\infty} (-)^r q^{r(r+1)} \sin\frac{2r+1}{2}(\pi i + \mathrm{a}).$$

The values of the functions $\vartheta_1, \vartheta_2, \zeta_3$ and ζ_4 for $v=0$ give rise (Prof. Cayley's memoir, page 912) to the constants,

$$k = \frac{\vartheta_2^2 0}{\vartheta_1^2 0}, \; k' = \frac{\vartheta_3^2 0}{\vartheta_1^2 0}, \; K = -\frac{\vartheta_1 0}{\vartheta_2 0}\cdot\frac{\vartheta_4' 0}{\vartheta_3 0}$$

or

$$k = 4\sqrt{q}\left\{\frac{1+q^2+q^6+\dots}{1+2q+2q^4+\dots}\right\}^2, = 4\sqrt{q}(1-4q+14q^2+\dots),$$

$$k' = \left\{\frac{1-2q+2q^4-\dots}{1+2q+2q^4+\dots}\right\}^2, = 1-8q+32q^2-96q^3+\dots,$$

$$K = \frac{\pi(1+2q+2q^4+\dots)(1-3q^2+5q^6-\dots)}{2(1-2q+2q^4-\dots)(1+q^2+q^6+\dots)}, = \tfrac{1}{2}\pi(1+4q+4q^2+\dots),$$

and identically $\quad k^2 + k'^2 = 1.$

These are the ordinary constants of elliptic functions.

Similar relations exist for the ξ, η and ζ groups. Taking first the ξ-groups and making $v=0$ we have

$$q^{\frac{1}{4}}\xi_1(0) = 1 + 2q\cos \mathrm{a} + 2q^4\cos 2\mathrm{a} + 2q^9\cos 3\mathrm{a} + \ldots, = A_1,$$
$$q^{\frac{1}{4}}\xi_2(0) = 2q^{\frac{1}{4}}\cos\tfrac{1}{2}\mathrm{a} + 2q^{\frac{9}{4}}\cos\tfrac{3}{2}\mathrm{a} + 2q^{\frac{25}{4}}\cos\tfrac{5}{2}\mathrm{a} + \ldots, = A_2,$$
$$e^{\frac{\pi}{2}}q^{\frac{1}{4}}\xi_3(0) = 1 - 2q\cos \mathrm{a} + 2q^4\cos 2\mathrm{a} - 2q^9\cos 3\mathrm{a} + \ldots, = A_3,$$
$$e^{\frac{\pi}{2}}q^{\frac{1}{4}}\xi_4(0) = -2q^{\frac{1}{4}}\sin\tfrac{1}{2}\mathrm{a} + 2q^{\frac{9}{4}}\sin\tfrac{3}{2}\mathrm{a} - 2q^{\frac{25}{4}}\sin\tfrac{5}{2}\mathrm{a} + \ldots, = A_4.$$

If here a is real (and therefore negative), and if we write $\mathrm{a} = -2\pi$, the series obviously take the same form as those for $\vartheta(0)$; but without making any hypothesis as to the value of a if we write

$$h = \frac{A_2^2}{\sqrt{A_1^4 + A_4^4}}, \quad h' = \frac{A_3^2}{\sqrt{A_1^4 + A_4^4}},$$

we find readily the relation $\quad h^2 + h'^2 = 1,$

that is $\quad A_2^4 + A_3^4 = A_1^4 + A_4^4,$

or $\quad q\xi_2^4(0) + e^{2\pi}q\xi_3^4(0) = q\xi_1^4(0) + e^{2\pi}q\xi_4^4(0),$

or $\quad \xi_2^4(0) + e^{2\pi}\xi_3^4(0) = \xi_1^4(0) + e^{2\pi}\xi_4^4(0).$

Taking now the η-groups and making $v=0$ and we have

$$\eta_1(0) = 1 + 2q\cosh\pi + 2q^4\cosh 2\pi + 2q^9\cosh 3\pi + \ldots,$$
$$\eta_2(0) = 2q^{\frac{1}{4}}\cosh\tfrac{1}{2}\pi + 2q^{\frac{9}{4}}\cosh\tfrac{3}{2}\pi + 2q^{\frac{25}{4}}\cosh\tfrac{3}{2}\pi,$$
$$\eta_3(0) = 1 - 2q\cosh\pi + 2q^4\cosh 2\pi - 2q^9\cosh 3\pi,$$
$$\eta_4(0) = i\{-2q^{\frac{1}{4}}\sinh\tfrac{1}{2}\pi + 2q^{\frac{9}{4}}\sinh\tfrac{3}{2}\pi - 2q^{\frac{25}{4}}\sinh\tfrac{5}{2}\pi + \ldots\}$$

Writing
$$g = \frac{\eta_2^2(0)}{\sqrt{\eta_1^4(0) + \eta_4^4(0)}}, \quad g' = \frac{\eta_3^2(0)}{\sqrt{\eta_1^4(0) + \eta_4^4(0)}},$$

we have $\quad g^2 + g'^2 = 1,$

or $\quad \eta_2^4(0) + \eta_3^4(0) = \eta_1^4(0) + \eta_4^4(0).$

Finally, taking the ζ-groups and making $v=0$ we have

$$q^{\frac{1}{4}}\zeta_1(0) = 1 + 2q\cos(\mathrm{a} + \pi i) + 2q^4\cos 2(\mathrm{a} + \pi i) + \ldots, = B_1,$$
$$q^{\frac{1}{4}}\zeta_2(0) = 2q^{\frac{1}{4}}\cos\tfrac{1}{2}(\mathrm{a} + \pi i) + 2q^{\frac{9}{4}}\cos\tfrac{3}{2}(\mathrm{a} + \pi i) + \ldots, = B_2,$$
$$e^{\frac{\pi}{2}}q^{\frac{1}{4}}\zeta_3(0) = 1 - 2q\cos(\mathrm{a} + \pi i) + 2q^4\cos 2(\mathrm{a} + \pi i) - \ldots, = B_3,$$
$$e^{\frac{\pi}{2}}q^{\frac{1}{4}}\zeta_4(0) = -2q^{\frac{1}{4}}\sin\tfrac{1}{2}(\mathrm{a} + \pi i) + 2q^{\frac{9}{4}}\sin\tfrac{3}{2}(\mathrm{a} + \pi i) - \ldots, = B_4.$$

Writing
$$F = \frac{B_2^2}{\sqrt{B_1^4 + B_4^4}}, \quad F' = \frac{B_3^2}{\sqrt{B_1^4 + B_4^4}},$$

we obtain $\quad F^2 + F'^2 = 1,$

or $\quad \zeta_2^4(0) + e^{2\pi}\zeta_3^4(0) = \zeta_1^4(0) + e^{2\pi}\zeta_4^4(0).$

On the Propagation of an Arbitrary Electro-Magnetic Disturbance, on Spherical Waves of Light and the Dynamical Theory of Diffraction.

By Professor H. A. Rowland.

INTRODUCTION.

In the year 1849 the great paper of Stokes "On the Dynamical Theory of Diffraction" was read before the Cambridge Philosophical Society, and this has remained until the present day the standard upon this important subject.

The method of Stokes was based upon the old elastic solid theory of light, and gave the following conclusions:

First. That when the incident light was plane polarized, the diffracted light from a small orifice was also plane polarized in such a manner that the displacement was in the same plane as that of the medium at the orifice. So that if a sphere was drawn with the orifice as a center and meridians drawn on the sphere with the axis in the direction of the vibration at the orifice, then these meridians represented the direction of displacement in the diffracted light.

Second. The intensity of the polarized light was represented as follows: Let δ represent the angle between the incident ray prolonged and the diffracted ray, and let ϕ be the angle between the diffracted ray and the direction of the displacement at the orifice. Then the intensity of the diffracted light around a very small orifice will be proportional to

$$(1 + \cos\delta)^2 \sin^2\phi .$$

The presence of the term in ϕ indicates that the intensity of the diffracted light at a given point varies as one rotates the plane of polarization. In ordinary light the intensity varies as $1 + \cos^2\delta$.

Stokes and others have attempted to determine the relation between the plane of polarization and the direction of displacement by means of the first relation, but they have not agreed with one another, and this want of agreement

has usually been assigned to the fact that the gratings used have been ruled on glass rather than in free space, as the equations indicate.

On examining this question from the point of view of the electro-magnetic theory of light, I have been led to entirely different results. But as the elastic solid theory for an incompressible solid must agree with the electro-magnetic theory, I have been led to examine the theory of Stokes, and believe that I have now discovered an error, which, if it were corrected, would lead to my result.

The results which I reach are as follows:

First. The plane of polarization of the diffracted light is determined as follows: Draw a sphere around the orifice and mark the point on the sphere where the incident light *enters* it. Through this point draw circles on the sphere whose planes are parallel to the electrostatic displacement at the orifice, and these circles give the direction of the electrostatic disturbance in the diffracted beam. Repeat the same for the magnetic disturbance, which is at right angles to the electrical disturbance, and the circles indicate the direction of the magnetic disturbance in the diffracted beam. These two systems of circles are orthogonal to one another, as they should be.

Second. The *intensity* of the diffracted light around a very small orifice *is symmetrical around the incident ray prolonged* and is proportional to

$$(1+\cos\delta)^2,$$

and the same expression applies to ordinary light.

Although the theory of diffraction forms the most interesting part of my paper, yet I have thought it worth while to treat of the general problem of spherical waves of light, which I have not seen considered anywhere else. The method is similar to that used in sound and the theory of heat conduction.

SPHERICAL WAVES.

Let F_0, G_0, H_0; F_1, G_1, etc., be a system of vectors derived from one another by the equations

$$F_{m+1}=\frac{dH_m}{dy}-\frac{dG_m}{dz}$$

$$G_{m+1}=\frac{dF_m}{dz}-\frac{dH_m}{dx}$$

$$H_{m+1}=\frac{dG_m}{dx}-\frac{dF_m}{dy}.$$

Then, as is well known, all this system of vectors necessarily satisfy the equation of continuity

$$\frac{dF_m}{dx} + \frac{dG_m}{dy} + \frac{dH_m}{dz} = 0,$$

except the primitive ones F_0, G_0, H_0, and even these can be made to satisfy the equation by the addition of terms of the form $\frac{dJ}{dx}$, $\frac{dJ}{dy}$, $\frac{dJ}{dz}$.

The equations of light-waves either on the electro-magnetic theory or the elastic solid theory are of the form

$$\frac{d^2F}{dt^2} = V^2\left\{\Delta^2 F - \frac{dJ}{dx}\right\} + v^2\frac{dJ}{dx}$$
$$\frac{d^2G}{dt^2} = V^2\left\{\Delta^2 G - \frac{dJ}{dx}\right\} + v^2\frac{dJ}{dy}$$
$$\frac{d^2H}{dt^2} = V^2\left\{\Delta^2 H - \frac{dJ}{dz}\right\} + v^2\frac{dJ}{dz}$$
$$J = \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}$$
$$\Delta^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}.$$

In the electro-magnetic theory v is always zero. In the elastic solid theory the terms in v give the wave of normal disturbance like that of sound, and can nearly always be omitted, as the transverse and normal waves are immediately separated when v is different from V, as it always must be. As we wish to use the derived functions only, the terms in J can also be omitted, as Maxwell has shown, and hence we can write on either theory

$$\frac{d^2F}{dt^2} = V^2\Delta^2 F$$
$$\frac{d^2G}{dt^2} = V^2\Delta^2 G$$
$$\frac{d^2H}{dt^2} = V^2\Delta^2 H$$

V indicates the velocity of a plane wave in the medium.

Each one of these equations is the same as that of sound and can be solved in a similar manner. If we have solutions of these equations F_0, G_0, H_0, which do not satisfy the equation of continuity, then we can get other solutions F_m, G_m, H_m from these which *shall* satisfy this condition. Denote $\sqrt{-1}$ by i and let $V_{-(n+1)}$ be a solid harmonic. Then as $V_{-(n+1)}$ is a homogeneous function

of x, y and z of the degree $-(n+1)$, we have

$$x\frac{dV_{-(n+1)}}{dx}+y\frac{dV_{-(n+1)}}{dy}+z\frac{dV_{-(n+1)}}{dz}=-(n+1)V_{-(n+1)}$$

and $$\Delta^2 V_{-(n+1)}=0.$$

Make $$F_0=C_n\rho^n V_{-(n+1)}\varepsilon^{(a-ib)(\rho-Vt)},$$

where C is a function of ρ of a complex form

$$C_n=A_n-iB_n$$
$$\rho^2=x^2+y^2+z^2.$$

This form has the advantage that by the addition of another term of the same form in which i has a negative value, the sum will reduce to a real form with circular arcs instead of powers of ε; a is introduced for generality. Substituting this form we have for the determination of C,* the equation

$$\frac{d^2C_n}{d\rho^2}+2(a-ib)\frac{dC_n}{d\rho}-\frac{n(n+1)}{\rho^2}C_n=0.$$

Writing $c=a-ib$ we readily find, as others have found before for this equation,

$$C_n=C_0\left\{1-\frac{n}{2}\frac{n+1}{c\rho}+\frac{n(n^2-1^2)}{2.4}\frac{n+2}{c^2\rho^2}-\frac{n(n^2-1^2)(n^2-2^2)}{2.4.6}+\text{etc.}\right\}$$

The series ends after $n+1$ terms, and so we can put it in the form

$$C_n=(-1)^n(1.3.5\ldots(2n-1))\frac{C_0}{c^n\rho^n}\left\{1-c\rho+\frac{2}{1.2}\frac{n-1}{2n-1}c^2\rho^2\right.$$
$$\left.-\frac{2.2}{1.2.3}\frac{(n-1)(n-2)}{(2n-1)(2n-2)}c^3\rho^3+\text{etc.}\right\}$$

The following properties of the functions C_n are useful. Writing $c=a-ib$, we have

$$C_n=\frac{c\rho}{2n+1}\{C_{n-1}-C_{n+1}\}$$

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* Making $$C_n=D_n\rho^{\frac{1}{2}}\varepsilon^{-\rho(a-ib)},$$
we obtain $$\frac{d^2D_n}{d\rho^2}+\frac{1}{\rho}\frac{dD_n}{d\rho}-D_n\left((a-ib)^2+\frac{(n+\frac{1}{2})^2}{\rho^2}\right)=0.$$

If we replace ρ by a new variable equal to $i(a-ib)\rho$, this reduces to the equation of Bessel's functions: hence we can write $$D_n=J_{(n+\frac{1}{2})}(b+ia)\rho,$$
$$F_0=\{J_{(n+\frac{1}{2})}(b+ia)\rho\}\rho^{n+\frac{1}{2}}V_{-(n+1)}\varepsilon^{-Vt(a-ib)},$$
with similar terms for G_0 and H_0.

Hence $$C_n=C_0'\rho^{\frac{1}{2}}\varepsilon^{-\rho(a-ib)}J_{(n+\frac{1}{2})}(b+ia)\rho,$$
where C_0' is a function of n to be determined, and one must add a corresponding term with $-i$ in place of $+i$.

$$\frac{dC_n}{d\rho} = c\left\{-C_n + \frac{1}{2n+1}[(n+1)C_{n-1} + nC_{n+1}]\right\} = cC_{n-1} - C_n\left(\frac{n}{\rho} + c\right),$$

$$C_{n+1} = C_n\left(1 - \frac{n+1}{c\rho}\right) + \frac{1}{c}\frac{dC_n}{d\rho}.$$

Putting $c\rho = cp\sqrt{1+2s}$ we have

$$\frac{d^{n+1}}{ds^{n+1}}\varepsilon^{c\rho} = \frac{c^{n+1}p^{2n+2}}{\rho^{n+1}}C_n\varepsilon^{c\rho}.$$

Extending the notation so that we should write $C_n(\rho)$ in the place of C_n as above, we have, if we put $pcs = q$

$$\varepsilon^{c\rho} = \varepsilon^{cp}\left\{1 + \frac{q}{1}C_0 + \frac{q^2}{1.2}C_1(p) + \frac{q^3}{1.2.3}C_2(p) + \text{etc.}\right\}$$

$$\frac{C_n(\rho)}{\rho^{n+1}}\varepsilon^{c\rho} = \frac{\varepsilon^{cp}}{p^{n+1}}\left\{C_n(p) + \frac{q}{1}C_{n+1}(p) + \frac{q^2}{1.2}C_{n+2}(p) + \text{etc.}\right\}$$

These expansions are useful in many calculations from the equations, and I believe they are given here for the first time, together with the differential expression from which they are derived.

Hence we have obtained a complete solution of one of the equations. Now the general value of $V_{-(n+1)}$ is

$$V_{-(n+1)} = (-1)^n\frac{d^n}{dh_1, dh_2, \ldots dh_n}\frac{1}{\rho}.$$

By varying the axes h_1, h_2, h_3, etc., the value and form of $V_{-(n+1)}$ will change, and can thus have many values while C remains unaltered. Select three of these forms and we can thus write

$$F_0 = C_n\rho^n V'_{-(n+1)}\varepsilon^{c(\rho - Vt)} = \frac{C_n}{\rho}Y'_n\varepsilon^{c(\rho - Vt)},$$

$$G_0 = C_n\rho^n V''_{-(n+1)}\varepsilon^{c(\rho - Vt)} = \frac{C_n}{\rho}Y''_n\varepsilon^{c(\rho - Vt)},$$

$$H_0 = C_n\rho^n V'''_{-(n+1)}\varepsilon^{c(\rho - Vt)} = \frac{C_n}{\rho}Y'''_n\varepsilon^{c(\rho - Vt)},$$

where Y'_n, Y''_n, Y'''_n are surface harmonics.

These do not satisfy the equation of continuity, but the series of derived vectors F_m, G_m, H_m satisfy both the equation of continuity and the equations of light.

One of the best forms for these spherical harmonics is to assume one of the axes in each of them parallel to one of the coordinate axes. We can then write

$$F_0 = C_n\rho^n\frac{dV_{-n}}{dx}\varepsilon^{c(\rho - Vt)},$$

$$G_0 = C_n\rho^n\frac{dV_{-n}}{dy}\varepsilon^{c(\rho - Vt)},$$

$$H_0 = C_n\rho^n\frac{dV_{-n}}{dz}\varepsilon^{c(\rho - Vt)}.$$

In this case the vector represented by the components F_0, G_0, H_0 is perpendicular to the surface $V_{-n} =$ constant.

Now the components of this vector do not satisfy the equation of continuity, and so, although it may represent the vector potential on the electro-magnetic theory, it cannot represent either the electric current or magnetic force, and is almost without direct use on the elastic solid theory. Hence the derived vectors are of more general use. These are

$$F_1 = cC_{n-1}\rho^{n-1}\left\{y\frac{dV_{-n}}{dz} - z\frac{dV_{-n}}{dy}\right\}\varepsilon^{c(\rho - Vt)},$$

$$G_1 = cC_{n-1}\rho^{n-1}\left\{z\frac{dV_{-n}}{dx} - x\frac{dV_{-n}}{dz}\right\}\varepsilon^{c(\rho - Vt)},$$

$$H_1 = cC_{n-1}\rho^{n-1}\left\{x\frac{dV_{-n}}{dy} - y\frac{dV_{-n}}{dx}\right\}\varepsilon^{c(\rho - Vt)}.$$

$$F_2 = -c^2F_0 + \frac{dJ_0}{dx},$$

$$G_2 = -c^2G_0 + \frac{dJ_0}{dy},$$

$$H_2 = -c^2H_0 + \frac{dJ_0}{dz}.$$

Where
$$J_0 = \frac{dF_0}{dx} + \frac{dG_0}{dy} + \frac{dH_0}{dz} = -\frac{n}{\rho}V_{-n}\frac{d}{d\rho}\{C_n\rho^n\varepsilon^{c(\rho - Vt)}\},$$

$$J_0 = -ncC_{n-1}\rho^{n-1}V_{-n}\varepsilon^{c(\rho - Vt)},$$

$$F_3 = -c^2F_1, \text{ etc.}$$

The remaining derived vectors are simply repetitions of these with only change in the constants.

These derived vectors can represent the vector potential, the electric displacement, the electric current, the magnetic displacement, or the magnetic force in the electro-magnetic theory, and the displacement, the velocity of the particles or the rotational displacement or velocity of rotation on the elastic solid theory.

The most interesting case is that of symmetry around an axis, say X. In this case
$$V_{-n} = \frac{d^n}{dx^n}\frac{1}{\rho} = \frac{1}{\rho^n}\frac{(-1)^n}{1.2.3\ldots n}Q_{n-1},$$
where Q_n is a zonal surface harmonic.

If α is the angle measured around the axis of x from y and θ that made with the same axis, and we put $\rho\sin\theta = r$,

we can then write, when m is even

$$R_m = \sqrt{G_m^2 + H_m^2},$$
$$N_{m+1} = \sqrt{G_{m+1}^2 + H_{m+1}^2},$$
$$M_{m+1} = N_{m+1} r$$

$$F_m = -\Delta^2 F_{m-2} = +\frac{1}{r}\frac{dM_{m-1}}{dr},$$
$$R_m = -\Delta^2 R_{m-2} - \frac{R_{m-2}}{r^2} = -\frac{1}{r}\frac{dM_{m-1}}{dx},$$

$$F_{m+1} = 0,$$
$$N_{m+1} = -\frac{dF_m}{dr} + \frac{dR_m}{dx} = -\Delta^2 N_{m-1} + \frac{N_{m-1}}{r^2} = -\frac{1}{r}\left\{\frac{d^2 M_{m-1}}{dx^2} + \frac{d^2 M_{m-1}}{dr^2} - \frac{1}{r}\frac{dM_{m-1}}{dr}\right\}.$$

The equation of continuity becomes

$$\frac{d(F_m r)}{dx} + \frac{d(R_m r)}{dr} = 0.$$

These can be expressed in terms of ρ and θ, as follows:

Let Θ_m, P_m and N_m be the components in the direction θ, ρ and α respectively. Then we have

$$\frac{d}{dr} = \sin\theta\frac{d}{d\rho} - \frac{\cos\theta\sin\theta}{\rho}\frac{d}{d\mu},$$
$$\frac{d}{dx} = \cos\theta\frac{d}{d\rho} + \frac{\sin^2\theta}{\rho}\frac{d}{d\mu},$$

$$\Theta_m = -F_m\sin\theta + R_m\cos\theta,$$
$$P_m = F_m\cos\theta + R_m\sin\theta,$$

$$\Theta_m = -\frac{1}{\rho\sin\theta}\frac{dM_{m-1}}{d\rho},$$
$$P_m = -\frac{1}{\rho^2}\frac{dM_{m-1}}{d\mu},$$
$$N_m = 0,$$

$$\Theta_{m+1} = 0,$$
$$P_{m+1} = 0,$$
$$N_{m+1} = -\frac{1}{\rho\sin\theta}\left\{\frac{d^2 M_{m-1}}{d\rho^2} + \frac{\sin^2\theta}{\rho^2}\frac{d^2 M_{m-1}}{d\mu^2}\right\}.$$

The equation of motion of light becomes in this case

$$\frac{1}{V^2}\frac{d^2 M_{m-1}}{dt^2} = \frac{d^2 M_{m-1}}{d\rho^2} + \frac{\sin^2\theta}{\rho^2}\frac{d^2 M_{m-1}}{d\mu^2},$$

whence
$$M_{m+1} = N_{m+1}\,\rho \sin\theta = -\frac{1}{V^2}\frac{d^2 M_{m-1}}{dt^2}.$$
Now, in this case of symmetry we readily find
$$F_0 = C_n \rho^n \frac{dV_{-n}}{dx}\,\varepsilon^{c(\rho - Vt)},$$
$$R_0 = C_n \rho^n \frac{dV_{-n}}{dr}\,\varepsilon^{c(\rho - Vt)},$$
and for convenience put $U = C_n \rho^n \varepsilon^{c(\rho - Vt)}$,

we readily find
$$M_1 = N_1 r = \rho \sin\theta \frac{dU}{d\rho}\left\{\sin\theta \frac{dV_{-n}}{dx} - \cos\theta \frac{dV_{-n}}{dr}\right\}.$$
Putting
$$Q'_{n-1} = \frac{dQ_{n-1}}{d\mu},$$
and, omitting the constant term, we have
$$M_1 = \frac{dU}{d\rho}\frac{1}{\rho^n}\sin^2\theta\, Q'_{n-1}.$$
Hence we have, by replacing $n-1$ by n, and changing the sign of all the terms,
$$M_1 = -C_n \sin^2\theta\, Q'_n\, \varepsilon^{c(\rho - Vt)}.$$
From which we find, writing P for the radial component,
$$\Theta_1 = 0,$$
$$P_1 = 0,$$
$$N_1 = -\frac{C_n}{\rho}\sin\theta\, Q'_n\, \varepsilon^{c(\rho - Vt)},$$
$$\Theta_2 = \frac{c}{\rho(2n+1)}\{(n+1)C_{n-1} + nC_{n+1}\}\sin\theta\, Q'_n\, \varepsilon^{c(\rho - Vt)},$$
$$P_2 = -\frac{n(n+1)}{\rho^2} C_n\, Q_n\, \varepsilon^{c(\rho - Vt)},$$
$$N_2 = 0,$$
$$N_3 = -c^2 N_1.$$
We can also go backward and write
$$\Theta_0 = -\frac{\Theta_2}{c^2},$$
$$P_0 = -\frac{P_2}{c^2},$$
$$M_{-1} = -\frac{M_1}{c^2}.$$

Hence, as I have remarked before, each of the vectors indicated by these components can be made equal to some one of the vectors used in the theory of light. Thus

ELECTRO-MAGNETIC THEORY.	ELASTIC SOLID THEORY.
Vector potential.	Linear displacement of particle.
Electric current or electro-motive force.	Rotation of particle.
Magnetic force, or induction, or magneto-motive force.*	Linear velocity of particle.
Rate of change of current, etc.	Angular velocity of particle, etc.
Rate of change of magnetic-force, etc.	

Should we add another term with $-i$ in the place of $+i$, we shall have the following real forms, since

$$C_n = A_n - iB_n,$$
$$C'_n = A_n + iB_n,$$
$$c = a - ib,$$
$$c' = a + ib.$$

$$N_1 = -\frac{2\sin\theta Q'_n}{\rho}\varepsilon^{a(\rho - Vt)}\{A_n \cos b(\rho - Vt) - B_n \sin b(\rho - Vt)\},$$

$$\Theta_2 = \frac{2\sin\theta Q'_n}{\rho}\varepsilon^{a(\rho - Vt)}\left\{\left[A_{n-1} - \frac{nA_n}{\rho}\right]\cos b(\rho - Vt) - \left[B_{n-1} - \frac{nB_n}{\rho}\right]\sin b(\rho - Vt)\right\},$$

$$P_2 = -\frac{2n(n+1)}{\rho^2}Q_n\varepsilon^{a(\rho - Vt)}\{A_n \cos b(\rho - Vt) - B_n \sin b(\rho - Vt)\}.$$

And the other terms are readily obtained from these as follows:

$$N_{-1} = \frac{2\sin\theta Q'_n}{\rho(a - ib)^2}\varepsilon^{a(\rho - Vt)}\{[(a^2 - b^2)A_n + 2abB_n]\cos b(\rho - Vt) - [(a^2 - b^2)B_n - 2abA_n]\sin b(\rho - Vt)\}$$

$$N_3 = \frac{2\sin\theta Q'_n(a - ib)^2}{\rho}\varepsilon^{a(\rho - Vt)}\{[(a^2 - b^2)A_n - 2abB_n]\cos b(\rho - Vt) - [(a^2 - b^2)B_n + 2abA_n]\sin b(\rho - Vt)\}.$$

As these quantities constantly return again to the same form with only a change in the constants, there will be only two cases, when the vector potential, or its analogy in the elastic solid theory, is made equal to the even or odd vectors. Let us take the first case, making the components of the vector potential equal to Θ_0 and P_0. The magnetic induction will then be N_1 and the components of the electric current

$$\frac{\Theta_2}{4\pi\mu} \text{ and } \frac{P_2}{4\pi\mu}.$$

* As Mr. Bosanquet has shown a disposition to claim this term, I may say that the idea expressed by it is of common occurrence in the works of Faraday, and I myself have used the term and given it mathematical expression in the American Journal of Mathematics. Mr. Bosanquet's article appeared in the Phil. Mag. for March, 1883.

Hence, in this case, the magnetic induction is in circles around the axis, and the electric currents in planes passing through the axis. At a great distance from the origin, the electric currents are on the sphere, but the normal component must still exist in order that the equation of continuity may be satisfied. The case of plane waves is the only one where the normal component entirely vanishes.

When we make the vector potential equal to N_1, the electric currents are in circles, and magnetic induction in planes passing through the axis.

Let us consider the first case. We shall find it much more convenient to continue with the imaginary form, remembering that a similar term with i changed in sign is always to be added except where it has already been done. We shall then have

Components of the Vector Potential

$$\Theta_0 = -\frac{\sin\theta\, Q_n'}{c(2n+1)\rho}\{(n+1)C_{n-1}+nC_{n+1}\}\varepsilon^{c(\rho-Vt)}.$$

$$P_0 = \frac{n(n+1)C_n}{c^2\rho^2}\,Q_n\,\varepsilon^{c(\rho-Vt)}.$$

Magnetic Induction

$$N_1 = -\frac{C_n}{\rho}\sin\theta\, Q_n'\,\varepsilon^{c(\rho-Vt)}.$$

Components of the Electric Current

$$\frac{\theta_2}{4\pi\mu} = \frac{c\sin\theta\, Q_n'}{4\pi\mu(2n+1)\rho}\{(n+1)(C_{n-1}) + n\,C_{n+1}\}\,\varepsilon^{c(\rho-Vt)},$$

$$\frac{P_2}{4\pi\mu} = -\frac{n(n+1)C_n}{4\pi\mu\rho^2}\,Q_n\,\varepsilon^{c(\rho-Vt)}.$$

Components of the Electric Displacement

$$-\frac{K}{4\pi}\frac{d\theta_0}{dt} = -\frac{VK\sin\theta\, Q_n'}{4\pi(2n+1)\rho}\{(n+1)\,C_{n-1} + n\,C_{n+1}\}\,\varepsilon^{c(\rho-Vt)},$$

$$-\frac{K}{4\pi}\frac{dP_0}{dt} = \frac{VKn(n+1)C_n}{4\pi c\rho^2}\,Q_n\,\varepsilon^{c(\rho-Vt)},$$

where μ is the magnetic permeability and K the inductive capacity, and we have the well-known relation of Maxwell,

$$V^2 = \frac{1}{\mu K}.$$

Should a perfectly conducting sphere exist, the electric density on its surface will be

$$\sigma = -\frac{K}{4\pi\rho^2}\frac{d}{dt}\frac{dM_{-1}}{d\mu} = -\frac{K}{4\pi}\frac{dP_0}{dt},$$

$$\sigma = \frac{VKn(n+1)C_n}{4\pi c\rho^2} Q_n \varepsilon^{c(\rho - Vt)}.$$

This is the same as the radial component of the electric displacement at this point.

Arbitrary Disturbance.

In the equations of p. 368 make $a = 0$ and $n = 1$. We then have

$$C_1 = C_0\left\{1 - \frac{i}{b\rho}\right\},$$

$$C_2 = C_0\left\{1 - \frac{3i}{b\rho} - \frac{3}{b^2\rho^2}\right\}.$$

The magnetic induction is then

$$N'' = -C_1\frac{\sin\theta}{\rho}\varepsilon^{-ib(\rho - Vt)},$$

and the components of the electric displacement,

$$\Theta' = -\frac{VK\sin\theta}{12\pi\rho}(2C_0 + C_2)\varepsilon^{-ib(\rho - Vt)}$$

$$P' = +\frac{iVK\cos\theta}{2\pi b\rho^2}C_1\varepsilon^{-ib(\rho - Vt)}.$$

Let now a sphere of radius R be circumscribed about the origin and an arbitrary uniform displacement take place in the interior of this sphere of a value equal to

$$X'\varepsilon^{-ib(R - Vt)} = C_0\frac{iVK}{2\pi bR^3}\varepsilon^{-ib(R - Vt)}\left\{1 - \frac{i}{bR}\right\}.$$

Taking this value for the displacement inside the sphere and the previous values for the outside, the equation of continuity is satisfied for the electric displacement and for the magnetic induction. If the sphere is very small indeed, we have

$$X' = \frac{iVKC_0}{2\pi b^2 R^3}.$$

Whence we have, on substituting the value of C_0 from this equation in the others and replacing $\frac{4}{3}\pi R^3$ by dv, the magnetic induction

$$N'' = -\frac{3b^2X'\sin\theta}{2VK\rho}\frac{C_1}{C_0}\varepsilon^{-ib(\rho - Vt)}dv,$$

and the electric displacement,

$$\Theta' = -\frac{2C_0 + C_2}{C_0}\frac{b^2X'}{8\pi\rho}\sin\theta\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$P' = \frac{C_1}{C_0}\frac{3ibX'}{4\pi\rho^2}\cos\theta\varepsilon^{-ib(\rho - Vt)}\,dv.$$

These equations give the complete solution of the problem of finding the disturbance at any point due to any arbitrary electrostatic disturbance throughout the medium. And it is to be noted that these equations are rigidly exact for all distances from the disturbance, and only have to be integrated to give the effect of any disturbance.

Had the disturbance been magnetic, we would have had for the magnetic induction,

$$\Theta'' = \frac{ib\sin\theta}{3\rho}(2C_0 + C_2)\,\varepsilon^{-ib(\rho - Vt)}$$

$$P'' = \frac{2\cos\theta}{\rho^2}C_1\varepsilon^{-ib(\rho - Vt)}.$$

And for the electric displacement,

$$N' = -\frac{iVKb\sin\theta}{4\pi\rho}C_1\varepsilon^{-ib(\rho - Vt)}.$$

Hence, taking a small sphere as before, the magnetic induction within it must be

$$X''\varepsilon^{-ib(R - Vt)} = \frac{2C_0}{R^2}\left(1 - \frac{i}{bR}\right)\varepsilon^{-ib(R - Vt)}$$

Whence, as before $$C_0 = -\frac{3b}{8\pi i}X''dv.$$

And so we have in this case for the magnetic induction

$$\Theta'' = -\frac{X''ib^2\sin\theta}{8\pi\rho}\frac{2C_0 + C_2}{C_0}\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$P'' = \frac{3X''ib\cos\theta}{4\pi\rho^2}\frac{C_1}{C_0}\varepsilon^{-ib(\rho - Vt)}\,dv,$$

and for the electric displacement

$$N' = \frac{3b^2VKX''\sin\theta}{32\pi^2\rho}\frac{C_1}{C_0}\varepsilon^{-ib(\rho - Vt)}.$$

These equations give the complete solution of the disturbance throughout the medium due to an arbitrary magnetic disturbance at any point.

Although the disturbance is harmonic, yet we know by Fourier's theorem that any disturbance can be represented by a series of harmonic terms with the

proper coefficients, and, indeed, we can replace the harmonic term by any function of $\rho - Vt$ or $\rho + Vt$. Should the disturbance not be parallel to the axis of X, we merely have to divide up the disturbance into its components and compute the effect of each and then add the components of the computed disturbance. In this way we readily find the effect of a general electric or magnetic disturbance.

Let the components of the arbitrary electric displacement be

$$X'\varepsilon^{ibVt}, \qquad X''\varepsilon^{ibVt},$$
$$Y'\varepsilon^{ibVt}, \text{ and of the magnetic induction } Y''\varepsilon^{ibVt},$$
$$Z'\varepsilon^{ibVt}, \qquad Z''\varepsilon^{ibVt},$$

where, in general, we must replace X', Y', Z', and X'', Y'', Z'' by a quantity of the complex form, and add another quantity with $-i$ in place of $+i$. Putting $D + iE$ for X', and adding the other term, we would have the real form

$$(D + iE)\varepsilon^{ibVt} + (D - iE)\varepsilon^{-ibVt} = 2\{D \cos bVt - E \sin bVt\},$$

which expresses the disturbance in any phase. But this is only necessary when we descend to actual calculation. The effect of this general disturbance is then found to be: The electric displacement

$$F' = \frac{b^2}{8\pi C_0\rho^3}\left\{X'(2C_0 + C_2)\rho^2 - 3S'C_2x - \frac{3KVC_1\rho}{4\pi}[Z''y - Y''z]\right\}\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$G' = \frac{b^2}{8\pi C_0\rho^3}\left\{Y'(2C_0 + C_2)\rho^2 - 3S'C_2y - \frac{3KVC_1\rho}{4\pi}[X''z - Z''x]\right\}\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$H' = \frac{b^2}{8\pi C_0\rho^3}\left\{Z'(2C_0 + C_2)\rho^2 - 3S'C_2z - \frac{3KVC_1\rho}{4\pi}[Y''x - X''y]\right\}\varepsilon^{-ib(\rho - Vt)}\,dv.$$

The magnetic induction,

$$F'' = \frac{b^2}{8\pi C_0\rho^3}\left\{X''(2C_0 + C_2)\rho^2 - 3S''C_2x + \frac{12C_1\pi\rho}{KV}[Z'y - Y'z]\right\}\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$G'' = \frac{b^2}{8\pi C_0\rho^3}\left\{Y''(2C_0 + C_2)\rho^2 - 3S''C_2y + \frac{12C_1\pi\rho}{KV}[X'z - Z'x]\right\}\varepsilon^{-ib(\rho - Vt)}\,dv,$$

$$H'' = \frac{b^2}{8\pi C_0\rho^3}\left\{Z''(2C_0 + C_2)\rho^2 - 3S''C_2z + \frac{12C_1\pi\rho}{KV}[Y'x - X'y]\right\}\varepsilon^{-ib(\rho - Vt)}\,dv,$$

where I have written

$$S' = X'x + Y'y + Z'z,$$
$$S'' = X''x + Y''y + Z''z.$$

For a general expression the term $\varepsilon^{-ib(\rho - Vt)}$ must be replaced by any function of $\rho - Vt$ or $\rho + Vt$ as before.

Before integration, one must of course substitute $(x - x')$, $(y - y')$ and $(z - z')$ for the x, y and z of the formula. It is evident that the components of the electric displacement can be replaced by the true convection displacement of

electricity as carried along by the actual motion of the medium, and the disturbance due to a moving magnet can be calculated in a similar manner.

To obtain an idea of the relative magnitude of the quantities which enter into these expressions, I may remark that the value of b in wave lengths is $\frac{2\pi}{\lambda}$. Hence, for any ordinary calculations with respect to light, the distance need be only a few inches, or, indeed, one inch, to cause the values of C_1, C_2, etc. to become constant and equal to C_0. But if we are treating of longer waves we must retain such terms.

I have already given the proper expansion into series of the quantities here involved, but it will be better to put them in a reduced form,

$$c = -ib; \quad \rho = p\sqrt{1+2s}; \quad cps = q.$$

$$C_n(p) = C_0\left\{1 - \frac{n}{2}\,\frac{n+1}{cp} + \frac{n(n^2-1)}{2.4}\,\frac{n+2}{c^2p^2} - \frac{n(n^2-1)(n^2-2)}{2.4.6}\,\frac{n+3}{c^3p^3} + \text{etc.}\right\}$$

$$\frac{C_1(\rho)\,\varepsilon^{c\rho}}{\rho^2} = \frac{\varepsilon^{cp}}{p^2}\left\{C_1(p) + \frac{q}{1}\,C_2(p) + \frac{q^2}{1.2}\,C_3(p) + \text{etc.}\right\}$$

$$\frac{C_2(\rho)\,\varepsilon^{c\rho}}{\rho^3} = \frac{\varepsilon^{cp}}{p^3}\left\{C_2(p) + \frac{q}{1}\,C_3(p) + \frac{q^2}{1.2}\,C_4(p) + \text{etc.}\right\}$$

$$\frac{2C_0 + C_2(\rho)}{\rho}\,\varepsilon^{c\rho} = \frac{3\varepsilon^{cp}}{p}\left\{\frac{1}{3}\left(2C_0 + C_2(p)\right) + \text{etc.}\right.$$
$$\left. + \frac{q^n}{(2n+3)1.2.3\ldots n}\left[(2n+2)C_n(p) + C_{n+2}(p)\right] + \text{etc.}\right\}$$

Thus one has $\quad \rho^2 = R^2 - 2(xx' + yy' + zz') + r^2,$

where $\quad R^2 = x^2 + y^2 + z^2$ and $r^2 = x'^2 + y'^2 + z'^2.$

Hence one can write $\quad p = R; \quad s = -\frac{xx' + yy' + zz'}{R^2} + \frac{r^2}{2R^2};$

$$q = -c\,\frac{xx' + yy' + zz'}{R} + c\,\frac{r^2}{2R}.$$

When R is great and r is small, the last term will vanish and leave a very convenient form for many cases.

But it is not necessary to always perform the calculation at a finite distance, for we have the following theorem which I believe to be new.

Theorem. Supposing the sources of light to be continuous and knowing the light over a sphere at an infinite distance, it can be determined for all space by the following process:

Let F, G and H be the components of any of the vectors such as the electric displacement, etc., and suppose them known over the infinite sphere.

Express them in terms of series of surface harmonics, thus:

$$F=\frac{C_0 \varepsilon^{c(\rho-Vt)}}{\rho}\{E_0 Y_0'+E_1 Y_1'+E_2 Y_2'+\text{ etc.}\}$$

$$G=\frac{C_0 \varepsilon^{c(\rho-Vt)}}{\rho}\{E_0 Y_0''+E_1 Y_1''+E_2 Y_2''+\text{ etc.}\}$$

$$H=\frac{C_0 \varepsilon^{c(\rho-Vt)}}{\rho}\{E_0 Y_0'''+E_1 Y_1'''+E_2 Y_2'''+\text{ etc.}\}$$

At any other point of space the equations must satisfy the equations of light motion and also the equation of continuity. Referring to the equations of pp. 363 and 364, we see that if we multiply each term by the corresponding quantity C_n, the equations of light will be satisfied, and if the surface harmonics are of the form

$$Y_n'=\rho^n\left\{y\frac{dV_{-(n+1)}}{dz}-z\frac{dV_{-(n+1)}}{dy}\right\},$$

$$Y_n''=\rho^n\left\{z\frac{dV_{-(n+1)}}{dx}-x\frac{dV_{-(n+1)}}{dz}\right\},$$

$$Y_n'''=\rho^n\left\{x\frac{dV_{-(n+1)}}{dy}-y\frac{dV_{-(n+1)}}{dx}\right\};$$

the equation of continuity is also satisfied. Hence the components of the vector at any point of space are

$$F=\frac{\varepsilon^{c(\rho-Vt)}}{\rho}\{C_0E_0 Y_0'+C_1E_1 Y_1'+C_2E_2 Y_2'+\text{ etc.}\}$$

$$G=\frac{\varepsilon^{c(\rho-Vt)}}{\rho}\{C_0E_0 Y_0''+C_1E_1 Y_1''+C_2E_2 Y_2''+\text{ etc.}\}$$

$$H=\frac{\varepsilon^{c(\rho-Vt)}}{\rho}\{C_0E_0 Y_0'''+C_1E_1 Y_1'''+C_2E_2 Y_2'''+\text{ etc.}\}$$

The best point for the origin will be somewhere near the center of gravity of the illuminated body.

That the light is perfectly determined in this way for all points outside a sphere around the origin which does not cut the illuminated body is evident from the fact that we might reverse the motions so as to make the light return along its previous path.

These expansions and this theorem entirely change the ideas of those who have only been in the habit of regarding light from the point of view of rays of light. For we here see that light coming from any source may be replaced by light from another source, so that the true source might be entirely invisible, and we "see" only the false source. This, however, could only happen for a point

outside a sphere drawn around the false point and the real point, and we could always detect the deception by making a complete spherical journey around the real point and inside the false point.

As an illustration, let us compute the effect of a circular electric current which is caused to vibrate back and forth according to a simple harmonic function, so as to make the displacement $C_0\varepsilon^{-cVt}$. In this case we must make

$$0 = X' = X'' = Y'' = Z''\,; \quad Y'dv = -C_0 r \sin \alpha d\alpha\,; \quad Z' = C_0 r \cos \alpha d\alpha\,,$$

where r is the radius of the circle.

From the symmetry around the axis of x we have $F' = 0$, and can write for the component of the electric displacement around x at a distance $R \sin\theta$ from that axis,

$$N' = -\frac{rb^2\varepsilon^{-cVt}}{8\pi}\int_0^{2\pi}\left\{\frac{2C_0 + C_2}{\rho}\sin\alpha - \frac{3RrC_2\sin\theta}{\rho^3}\cos^2\alpha\right\}\varepsilon^{c\rho}d\alpha\,,$$

$$\rho^2 = R^2 - 2Rr\sin\theta\sin\alpha + r^2.$$

Integrating the first term by parts, taking $\sin\alpha d\alpha$ for one part we have, since the first part disappears,

$$\int_0^{2\pi}\frac{2C_0 + C_2}{\rho}\varepsilon^{cs}\sin\alpha d\alpha = \int_0^{2\pi}\frac{d}{d\alpha}\left(\frac{2C_0 + C_2}{\rho}\varepsilon^{cs}\right)\cos\alpha d\alpha\,.$$

But
$$\frac{d}{d\alpha} = \frac{d\rho}{d\alpha}\frac{d}{d\rho} = -\frac{Rr\sin\theta\cos\alpha}{\rho}\frac{d}{d\rho}\,.$$

Whence
$$\frac{d}{d\alpha}\left(\frac{2C_0 + C_2}{\rho}\varepsilon^{c\rho}\right) = -Rr\sin\theta\cos\alpha\frac{3}{5}\frac{c\varepsilon^{c\rho}}{\rho^2}\left[4C_1 + C_3\right].$$

Whence we have

$$N' = \frac{3cRr^2b^2\sin\theta\varepsilon^{-cVt}}{8\pi}\int_0^{2\pi}\frac{C_1\varepsilon^{c\rho}}{\rho^2}\cos^2\alpha d\alpha\,.$$

In the expansion, put

$$p = R\,; \quad q = -cr\sin\theta\sin\alpha + c\frac{r^2}{2R}\,.$$

$$\frac{C_1\varepsilon^{c\rho}}{\rho_2} = \frac{\varepsilon^{cR}}{R^2}\left\{C_1(R) + \frac{q}{1}C_2(R) + \frac{q^2}{1.2}C_3(R) + \text{etc.}\right\}$$

Writing
$$g = \frac{cr^2}{2R} \text{ and } h = -cr\sin\theta\,,$$

we have
$$q = g + h\sin\alpha\,,$$

$$\int_0^{2\pi}q^n\cos^2\alpha d\alpha = 2\left\{\frac{1}{2}g^n + \frac{n(n-1)}{1.2}\frac{1}{2.4}g^{n-2}h^2 + \frac{n(n-1)(n-2)(n-4)}{1.2.3.4}\frac{1.3}{2.4.6}g^{n-4}h^4\right.$$
$$\left. + \text{etc.} + \frac{n(n-1)\ldots(n-2m+1)}{12\ldots 2m}\frac{1.3\ldots(2m-1)}{2.4\ldots(2m+2)}g^{n-2m}h^{2m} + \text{etc.}\right\}$$

So that the problem is completely solved without any approximation and for all distances at which the series is convergent.

At a great distance the expression becomes very simple.

$$N' = \frac{3cr^2b^2C_0}{4\pi}\frac{\sin\theta}{R}\varepsilon^{c(R-Vt)}\left\{\frac{1}{2} + \frac{1}{2^2.4}\sigma^2r^2\sin^2\theta + \frac{1}{(2.4)^2 6}\sigma^4r^4\sin^4\theta \right.$$
$$\left. + \text{etc.} + \frac{1}{(2.4\ldots 2n)^2(2n+2)}c^{2n}r^{2n}\sin^{2n}\theta + \text{etc.}\right\}$$

We recognize this series as a Bessel's function divided by $br\sin\theta$, and so we can write

$$N' = -\frac{3ib^2rC_0}{4\pi R}\varepsilon^{c(R-Vt)}J_1(br\sin\theta).$$

This same series occurs in the expression for the light from a circular orifice, but I am not aware that writers on physical optics have recognized this connection with Bessel's functions. Prof. Stokes has given the value of the series for large values of $br\sin\theta$, and the same value is given by writers on Bessel's functions. We thus find, writing $v = br\sin\theta$

$$N' = -\frac{3ib^2rC_0}{4\pi R}\sqrt{\frac{2}{\pi v}}\,\varepsilon^{c(R-Vt)}\left[\begin{array}{l}\sin\left(v-\frac{\pi}{4}\right)\left\{1+\frac{3.5.1}{2.4}\frac{1}{(4v)^2}-\frac{3.5.7.9.1.3.5}{2.4.6.8}\frac{1}{(4v)^4}+\text{etc.}\right\}\\ +\cos\left(v-\frac{\pi}{4}\right)\left\{\frac{3}{2}\frac{1}{4v}-\frac{3.5.7.1.3}{2.4.6}\frac{1}{(4v)^3}+\text{etc.}\right\}\end{array}\right]$$

The energy given out per unit of time is, by the first formula,

$$\frac{3Vr^4b^6C_0^2}{2K}\left\{1-\frac{b^2r^2}{5}+\frac{b^4r^4}{35}-\frac{b^6r^7}{720}+\frac{b^8r^8}{31680}-\frac{b^9r^9}{1310400}+\text{etc.}\right\}$$

and by the second for very large values of br

$$\frac{Vrb^3C_0^2}{2\pi K}.$$

Let us now reduce the series to spherical harmonics.

Write

$$u = cr = -ibr = -\frac{2\pi ir}{\lambda},$$

$$P = u^2\left\{\frac{1}{2}+\frac{u^2}{2^2 4}\sin^2\theta+\frac{u^4}{(2.4)^2 6}\sin^4\theta+\text{etc.}\right\}$$

This satisfies the differential equation

$$\frac{d^2P}{du^2}-\frac{1}{u}\frac{dP}{du}-P\sin^2\theta = 0.$$

Write also $\mu = \cos\theta$ and $Q'_n = \frac{dQ_n}{d\mu}$. P can be developed in the following series as it only contains the even powers of μ;

$$P = A'_1 + A'_3Q'_3 + A'_4Q'_4 + \text{etc.}$$

But

$$Q'_n \sin^2\theta = -\frac{n(n+1)}{(2n-1)(2n+1)(2n+3)}\{(2n-1)Q'_{n+2}-(4n+2)Q'_n+(2n+3)Q'_{n-2}\}.$$

Writing $\delta = \frac{d^2}{du^2} - \frac{1}{u}\frac{d}{du}$ we have

$$A'_3 = \frac{5.7}{3.4}\left\{\frac{4}{5}A'_1 - \delta A'_1\right\},$$

$$A'_5 = \frac{9.11}{5.6}\left\{2\frac{3.4}{5.9}A'_3 - \frac{2}{15}A'_1 - \delta A'_3\right\},$$

$$A'_7 = \text{etc.},$$

$$A'_n = \frac{(2n-1)(2n+1)}{n(n+1)}\left\{2\frac{(n-2)(n-1)}{(2n-5)(2n-1)}A'_{n-2} - \frac{(n-4)(n-3)}{(2n-5)(2n-7)}A'_{n-4} - \delta A'_{n-2}\right\}.$$

These all depend on A'_1, whose value can be found by developing $\sin^2\theta$, $\sin^4\theta$, etc., in harmonics. In the following values one must add the second term with $-i$ in place of $+i$ before substituting in the formula

$$A'_1 = \frac{3}{4}C_1(r)\,\varepsilon^{cr}$$

$$A'_3 = -\frac{21}{16}C_3(r)\,\varepsilon^{cr}$$

$$A'_5 = \frac{33}{10}\left\{\frac{1}{5}C_1(r) + \frac{5}{16}C_3(r)\right\}\varepsilon^{cr}$$

$$A'_7 = \text{etc.}$$

These reduce to Bessel's functions when we obtain the real form. Thus

$$A'_1 = \frac{3}{2}\left\{\frac{\sin br}{br} - \cos br\right\} = \frac{3}{2}\sqrt{\frac{\pi br}{2}}\,J_{\frac{3}{2}}(br).$$

Each term of the original series is now of the form $A'_n Q'_n \sin\theta$. But by the theorem of p. 372 we have only to multiply the terms of this by $C_n(R)$ and it gives the value for any finite distance outside a sphere around the origin which contains the circle. Hence

$$N' = \frac{3i\sin\theta}{4\pi R}\,\varepsilon^{c(R-Vt)}\{A'_1 C_0 + A'_3 Q'_3 C_3(R) + A'_5 Q'_5 C_5(R) + \text{etc.}\}.$$

DYNAMICAL THEORY OF DIFFRACTION.

When a ray of light strikes upon a screen in which there is an opening, a disturbance takes place in that opening, whose effect can be calculated by the preceding formulæ. Maxwell has shown, that in a plane wave, the energy is half magnetic and half electrostatic, and that the magnetic and electric displacements are at right angles to each other and in the same phase.

Stokes' solution is based upon the displacement and rate of displacement of his elastic medium. But in an elastic wave there is not only displacement but *rotation* also, and the components of this rotation must satisfy the equation of continuity. But when a wave is broken up at an orifice, the rotation is left discontinuous by Stokes' solution, and hence it cannot be exact. The equation of propagation of the rotation is the same as that of the displacement, and the two are at right angles to each other, and they are both equally important.

Hence, on the elastic solid theorg as well as the electro-magnetic theorg, the true solution of diffraction will depend upon the sums of two similar terms.

When we take account of both the terms, the fundamental properties by which Stokes attempted to obtain the direction of the motion of the particles vanishes, and that problem is impossible of solution by this means.

In the general equations of p. 371 let all the disturbances vanish except X' and Y'', so that the electric disturbance is in the direction of X and the magnetic in that of Y.

In order that the energy coming from the two disturbances may be equal we must have

$$\frac{V^2K}{8\pi}\left(\frac{3b^2X'}{2VK}\right)^2=\frac{V^2K}{8\pi}\left(\frac{3b^2Y''}{8\pi}\right)^2,$$

or

$$Y''=\frac{4\pi}{VK}X'$$

The electric displacement at any other point of space will then be

$$F'=\frac{b^2X'\nu}{8\pi C_0\rho^3}\{(2C_0+C_2)\rho^2-3C_2x^2+3C_1\rho z\}\,\varepsilon^{-ib(\rho-Vt)}ds,$$

$$G'=\frac{b^2X'\nu}{8\pi C_0\rho^3}\{-3C_2xy\}\,\varepsilon^{-ib(\rho-Vt)}ds,$$

$$H'=\frac{b^2X'\nu}{8\pi C_0\rho^3}\{-3C_2xz-3C_1x\rho\}\,\varepsilon^{-ib(\rho-Vt)}ds.$$

For showing the peculiarities of the case, polar coordinates are best. Let θ be the angle made by ρ with the axis of z and ϕ the angle around it from the plane XY. Let Θ', Θ'', Φ', Φ'', and P', P'' be the components of the electric displacement and magnetic induction to increase these angles and in the direction of ρ respectively. Then we have for the electric displacement

$$\Theta'=\frac{3b^2X'\nu}{8\pi\rho}\cos\phi\left\{[1+\cos\theta]\left[1-\frac{i}{b\rho}\right]-\frac{\cos\theta}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho-Vt)}ds,$$

$$\Phi'=-\frac{3b^2X'\nu}{8\pi\rho}\sin\phi\left\{[1+\cos\theta]\left[1-\frac{i}{b\rho}\right]-\frac{1}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho-Vt)}ds,$$

$$P'=\frac{3ibX'\nu}{4\pi\rho^2}\sin\theta\cos\phi\left\{1-\frac{i}{b\rho}\right\}\varepsilon^{-ib(\rho-Vt)}ds.$$

And for the magnetic induction,

$$\Theta'' = \frac{3b^2X'\nu}{2VK\rho}\sin\phi\left\{[1+\cos\theta]\left[1-\frac{i}{b\rho}\right]-\frac{\cos\theta}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho-Vt)}ds,$$

$$\Phi'' = \frac{3b^2X'\nu}{2VK\rho}\cos\phi\left\{[1+\cos\theta]\left[1-\frac{i}{b\rho}\right]-\frac{1}{b^2\rho^2}\right\}\varepsilon^{-ib(\rho-Vt)}ds,$$

$$P'' = \frac{3ibX'\nu}{VK\rho}\sin\theta\sin\phi\left\{1-\frac{i}{b\rho}\right\}\varepsilon^{-ib(\rho-Vt)}ds.$$

In these, ν is the thickness of the disturbed stratum, so that $\nu ds = dv$. The electric displacement within the stratum is $X'\varepsilon^{ibVt}$.

When $b\rho$ is very large, the disturbance is in the spherical surface, and indeed it only requires a fraction of an inch from the origin to be able to omit the terms in $b\rho$, and also P' and P''. The equations then become very simple, as follows. The electric displacement,

$$\Theta' = \frac{3b^2X'\nu}{8\pi\rho}\cos\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds,$$

$$\Phi' = \frac{3b^2X'\nu}{8\pi\rho}\sin\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds,$$

and the magnetic induction,

$$\Theta'' = \frac{3b^2X'\nu}{2VK\rho}\sin\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds,$$

$$P'' = \frac{3b^2X'\nu}{2VK\rho}\cos\phi\,(1+\cos\theta)\,\varepsilon^{-ib(\rho-Vt)}ds.$$

From either of these expressions or the more exact ones we see that the distribution of the magnetic induction is exactly the same, but turned around the axis of Z 90°, as the electric displacement. As this result would also apply to the elastic solid theory, we conclude that *diffraction gives no means of determining the relation between the displacement and plane of polarization.* Had we only taken the original disturbance as electrical alone, we would have arrived at Stokes' result.

Squaring the coefficients of the time function and adding, we find

$$\frac{9b^2X'^2\nu^2}{64\pi\rho^2}(1+\cos\theta)^2(ds)^2.$$

Hence the light is symmetrical around the axis of Z and varies from 4 in the positive direction to 1 in the plane XY and 0 in the negative direction.

It now remains to connect the arbitrary displacement $X'\nu$ with the intensity of the original wave.

Let the arbitrary displacement $X'\nu\varepsilon^{ibVt}$, with the corresponding magnetic quantity, exist throughout the plane XY. From considerations of symmetry, the electric displacement throughout space must be parallel to the axis X, and so we can write,

$$F' = \iint \frac{b^2 X'\nu}{8\pi C_0 \rho^3} \{(2C_0 + C_2)\rho^2 - 3C_2 x^2 + 3C_1 \rho z\} \varepsilon^{-ib(\rho - Vt)} ds,$$

$$z = \text{constant},$$

$$x = -r\cos\phi, \quad ds = rd\phi dr = \rho d\phi d\rho,$$

$$y = -r\sin\phi. \quad \rho^2 = z^2 + r^2.$$

The general and exact integral is

$$F' = \frac{3ibX'\nu}{8}\left\{1 + \frac{i + 2bz}{b\rho} + \frac{z^2}{\rho^2} - \frac{iz^2}{b\rho^3}\right\}\varepsilon^{-ib(\rho - Vt)}.$$

For positive values of z this gives, between the limits $\rho = \infty$ and $\rho = z$,

$$F = -\frac{3ibX'\nu}{2}\varepsilon^{-ib(z - Vt)}.$$

But for z negative it is zero. Hence such an arbitrary disturbance produces a wave in the positive direction, but none in the negative direction.

No approximation has been made in obtaining this quantity, and it evidently applies to a plane of any size, even infinitessimal, provided the point under consideration is infinitely near to it. The displacement near the surface therefore differs in phase $\frac{1}{2}\pi$ from the arbitrary disturbance, but is dependent upon its value at that particular point.

Hence we can replace any particular wave surface by a surface of arbitrary disturbance whose phase differs $\frac{1}{2}\pi$ from that of the wave. Such a surface of arbitrary disturbance then produces the same effect at all points of space as the original wave. Should the wave be spherical and of short radius, the normal component will enter and complicate the result, though the solution can be obtained but would evidently be complicated.

But if the wave have a radius of an inch or even less, the displacement is practically perpendicular to the radius, and the solution here given will apply.

Let l, m, n be the direction cosines of the normal to such a wave, and l', m' and n' the direction cosines of the electric displacement which is represented by

$$I\varepsilon^{-ib(R - Vt)}.$$

Then the values of the arbitrary displacement and magnetic induction to substi-

tute in the general equations of p. 371, to produce such a wave will evidently be

$$X'\nu = \frac{2i}{3b} Il', \quad X''\nu = \frac{8\pi i}{3VKb} Il'',$$

$$Y'\nu = \frac{2i}{3b} Im', \quad Y''\nu = \frac{8\pi i}{3VKb} Im'',$$

$$Z'\nu = \frac{2i}{4b} In', \quad Z''\nu = \frac{8\pi i}{3VKb} In'',$$

Where l'', m'', n'' are the direction cosines of the magnetic induction. We also have

$$ll' + mm' + nn' = 0,$$
$$ll'' + mm'' + nn'' = 0,$$
$$l'l'' + m'm'' + n'n'' = 0.$$

Returning again to the equations of p. 377, we see that the polarized light is diffracted equally in all directions from a very small orifice and independent of its plane of polarization. Furthermore the plane of polarization at any point is found by drawing a sphere through that point with its center at the orifice, and then drawing a plane through the given point and the point where the incident light first cuts the sphere, and cutting the plane of polarization of the incident light in a line perpendicular to the incident ray. The intersection of the plane and sphere then give the direction of the polarization.

It is seen that in both these particulars my solution differs from that of Prof. Stokes, and the construction is the same whether one takes the electric or magnetic quantities as the direction of polarization.

The system of planes for the electric and magnetic quantities form a system of orthogonal circles on the sphere.

The following constructions can also be used for obtaining the direction of the electric and magnetic quantities. Draw a sphere around the orifice and draw an axis through the sphere in the direction of the incident light. Then rule a sheet of paper with a series of lines at right angles to each other. Cut a star shaped piece out of the paper with its diameter equal to the circumference of the sphere, and having a very large number of points. Place the center of the star on the sphere at the end of the axis where the light leaves the sphere, and wrap the points around the sphere, the points meeting around the incident ray. The marks on the paper then give the required directions.

We can also construct the curves of polarization by noting that the stereographic projection of the lines on a plane is merely a series of straight lines.

The equations become very simple at many wave lengths distance from the orifice, especially when the orifice is small. If the radius of the original wave is large, it is usually sufficient to consider the periodic factor as the only variable. In this case we can write

$$\Theta' = \frac{ibI}{4\pi\rho} \cos\phi \left[1 + \cos\theta\right] \iint \varepsilon^{-ib(\rho - Vt)} ds,$$

$$\Phi' = -\frac{ibI}{4\pi\rho} \sin\phi \left[1 + \cos\theta\right] \iint \varepsilon^{-ib(\rho - Vt)} ds,$$

$$P' = 0,$$

where I is the coefficient of the original vibration, and therefore its square is the intensity of the original light. The intensity of the diffracted light is simply proportional to the sum of the squares of Θ' and Φ'. This is the expression ordinarily used except the term in θ. Thus for a circular orifice we have the vibration expressed in Bessel's functions,

$$\Theta' = \frac{iIr}{2R} \cos\phi (1 + \cos\theta) \frac{J_1(br \sin\theta)}{\sin\theta} \varepsilon^{-ib(R - Vt)},$$

$$\Phi' = -\frac{iIr}{2R} \sin\phi (1 + \cos\theta) \frac{J_1(br \sin\theta)}{\sin\theta} \varepsilon^{-ib(R - Vt)}.$$

It is impossible to pass from these expressions to the case of a plane wave since they are only for the case of a great distance from a small orifice.

The Method of Graphs applied to Compound Partitions.

By G. S. Ely, Ph.D., *Professor of Mathematics in Buchtel College.*

If we divide a number N into two portions in all possible ways, and then partition each of the portions in all possible ways, we shall form all the possible bi-partitions of N. If the number of portions is three we form all the tri-partitions of N; and if the number of portions is unlimited we shall form all the compound partitions of N. Let

$$a_1 a_2 \ldots a_\alpha \,|\, b_1 b_2 \ldots b_\beta \,|\, c_1 c_2 \ldots c_\gamma \,|\, \ldots \,|\, e_1 e_2 \ldots e_\epsilon$$

be such a compound partition of N. I shall call this a regular compound partition of N if the following conditions are satisfied:

$$a_i \nless b_i \nless c_i \ldots \nless e_i,$$

whatever i may be.

Now all such regular compound partitions may be represented graphically by an array of points in space as follows: let each of the portions be represented by an array of points in a plane and then let the planes be superimposed: the plane containing the first portion on top and that containing the second portion next, etc.; first lines lying above first lines and first columns above first columns. Then it is evident, that in general, any compound partition may be read in six different ways: that is, to any given compound partition there are five others which are conjugate. For example, the compound partitions conjugate to

531 | 211 | 11 are 32211 | 31 | 2; 521 | 311 | 11;
32111 | 311 | 2; 332 | 21 | 2 | 1 | 1 and 332 | 21 | 11 | 1 | 1.

The following method may be given for obtaining the five compound partitions which are the conjugates of any given compound partition: the first one of the conjugates is obtained by writing for each portion of the given compound partition its conjugate single partition—thus 32211 is the conjugate of 531: the second of the conjugates is obtained from the given compound partition by taking the first of the elements of each portion of the given compound partition to form the first portion of the new conjugate, and the second of the elements of each portion to form the second portion, etc.: the third and fifth conjugates

are obtained from the second and fourth conjugates respectively, in the same manner as the first conjugate was obtained from the given compound partition: finally, the fourth conjugate is obtained from the first conjugate in the same manner that the second conjugate was obtained from the given compound partition; or it may be obtained from the given compound partition directly by an extension of Professor Sylvester's method of calling off—thus 1 goes in 3 elements of the first portion of the given compound partition, in 3 elements of the second portion and in 2 elements of the third portion, giving 332 for the first portion of the fourth conjugate; again, 2 goes in 2 elements of the first portion and in 1 element of the second portion, giving 21 for the second portion of the fourth conjugate, etc.

The points lie in three sets of parallel planes. If the array of points in each of the planes of one of the three sets be symmetrical, then there will be three conjugates instead of six, as for example, the three following compound partitions are conjugates $531|311|11$; $32211|311|2$; $332|21|21|1|1$.

In addition there are the self-conjugate compound partitions in which the array of points in each of the planes of all three sets of parallel planes is symmetrical. Among the smaller numbers these are not very numerous: 1 has the self-conjugate partition 1; 4 has the self-conjugate compound partition $21|1$; and there are no more examples till we come to 7, which has the two self-conjugate compound partitions $311|1|1$ and $22|21$. And it is easy to see that in the cases where there is no cube equal to or less than the number to which the number is congruous with respect to the modulus 3, there can be no self-conjugate compound partition.* Thus the following numbers are void of self-conjugate compound partitions: 2, 3, 5, 6, 9, 12, 15, 18, 21, 24; and all other numbers have self-conjugate compound partitions.

If we use the symbol $(w;\ n;\ i,\ j,)$

to signify the number of ways in which we can divide the number w into regular compound partitions: the number of portions being not more than n, each portion being partitioned into i or fewer parts not greater than j, then we evidently have

$$(w;\ n;\ i,\ j,) = (w;\ n;\ j,\ i,) =$$
$$(w;\ i;\ n,\ j,) = (w;\ i;\ j,\ n,) =$$
$$(w;\ j;\ n,\ i,) = (w;\ j;\ i,\ n,)$$

* This is, it may be remarked, *en passant*, exactly similar to the case of single partitions in which there is no self-conjugate partition for a number n, if there be no square equal to or less than n, to which the number is congruous with respect to the modulus 2: and the number 2 is the only number fulfilling that condition.

Of this we may notice a special example, namely, if $n = 2$ and $i = j = w$, we have all the regular bi-partitions of w. Then we shall have $(w;\ 2;\ w,\ w,) = (w;\ w;\ 2,\ w,) = (w;\ w;\ w,\ 2,)$ that is, the number of regular bi-partitions of any number, w, is equal to the number of compound partitions of w in which the number of parts in which any portion is partitioned is not greater than two, and also to the number of compound partitions of w in which no part in any portion is greater than two.

It may also be noticed that we may have like results in the case of what I shall call regular n-compound partitions. For example, all the bi-compound partitions of any number N are obtained by writing all the single partitions of N and then partitioning each of the elements of each single partition of N into compound partitions. Then if

$$a_{11} a_{12} \ldots a_{1\alpha_1} | b_{11} \ldots b_{1\beta_1} | \ldots | e_{11} \ldots e_{1\epsilon_1} || a_{21} a_{22} \ldots a_{2\alpha_2} | b_{21} \ldots b_{2\beta_2} | \ldots$$
$$| e_{21} \ldots e_{2\epsilon_2} || \ldots || a_{j_1} a_{j_2} \ldots a_{j\alpha_j} | b_{j_1} \ldots b_{j\beta_j} | \ldots | e_{j_1} \ldots e_{j\epsilon_j}$$

be a bi-compound partition of N, it is termed regular if the following conditions are satisfied

$$a_{ik} \nless b_{ik} \nless c_{ik} \nless \ldots \nless e_{ik}$$

and $$d_{1n} \nless d_{2n} \nless d_{3n} \nless \ldots \nless d_{jn}, \text{ where}$$

i, k, n have any values and d means any of the letters a, b, $c \ldots e$. Then we can represent any such bi-compound partition by an array of points in four-fold space and accordingly have in general twenty-four conjugates, which may, in special cases, reduce to twelve or four. There will also be self-conjugate bi-compound partitions in the cases in which the number N is such that there is some number equal to or less than N which is the fourth power of some integer, to which fourth power N is congruous with respect to the modulus 4. For all numbers then of the form $4n + 2$ or $4n + 3$ and for the numbers 4, 8 and 12, there are no self-conjugate bi-compound partitions. If the symbol

$$(w;\ m;\ n;\ i,\ j,)$$

be used to signify the number of ways in which we can divide the number w into regular bi-compound partitions, the number of apportionments being not more than m, the number of portions in each apportionment not greater than n, each portion being partitioned into i or fewer parts not greater than j, then we have of course $$(w;\ m;\ n;\ i,\ j,) = (w;\ m_1;\ n_1;\ i_1,\ j_1,),$$
where the numbers m_1, n_1, i_1, j_1, are the numbers m, n, i, j, in any of the 24 possible orders of arrangement.